جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة الفرات الأوسط التقنية / كلية
تقنيات التاهيل الطبي والاطراف
الصناعية

الحقيبة التدريسية لمادة مبادئ الميكانيك المرحلة الاولى / قسم تقنيات الاطراف والمساند الطبية / الفصل الدراسي الثاني

اعداد

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دكتوراه في هندسة الميكانيك \
الاطراف الصناعية الذكية

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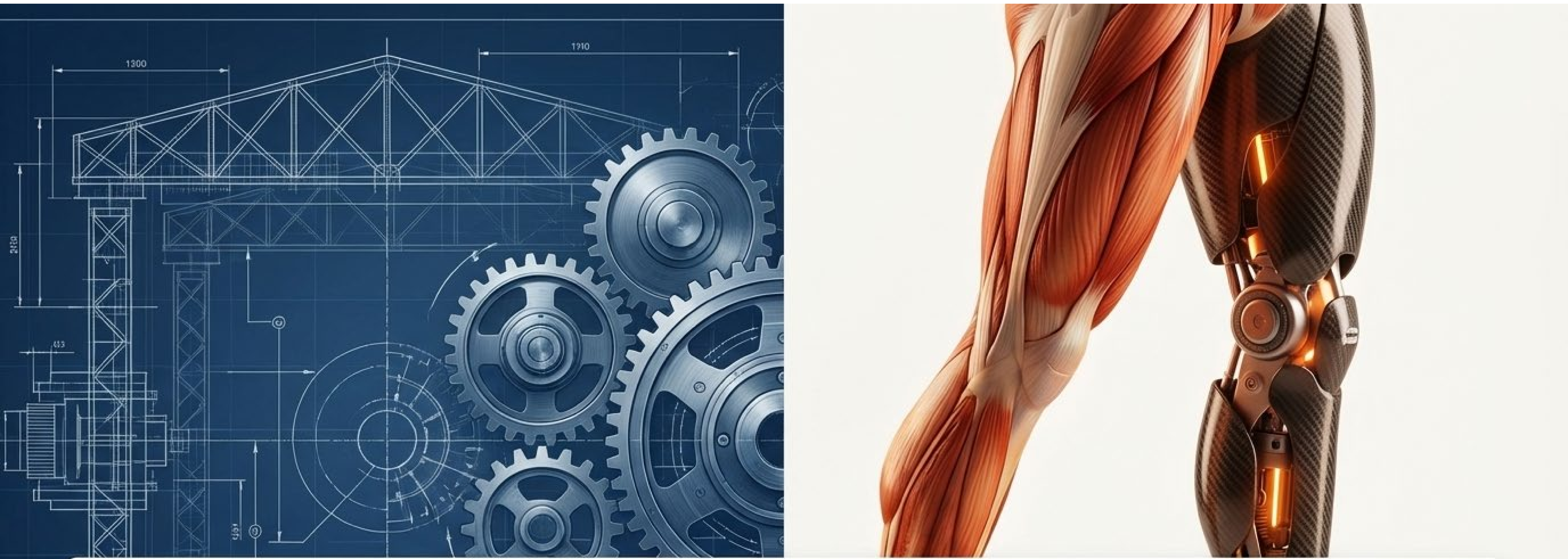
Lecture 1

Introduction of Princple of Mechanics

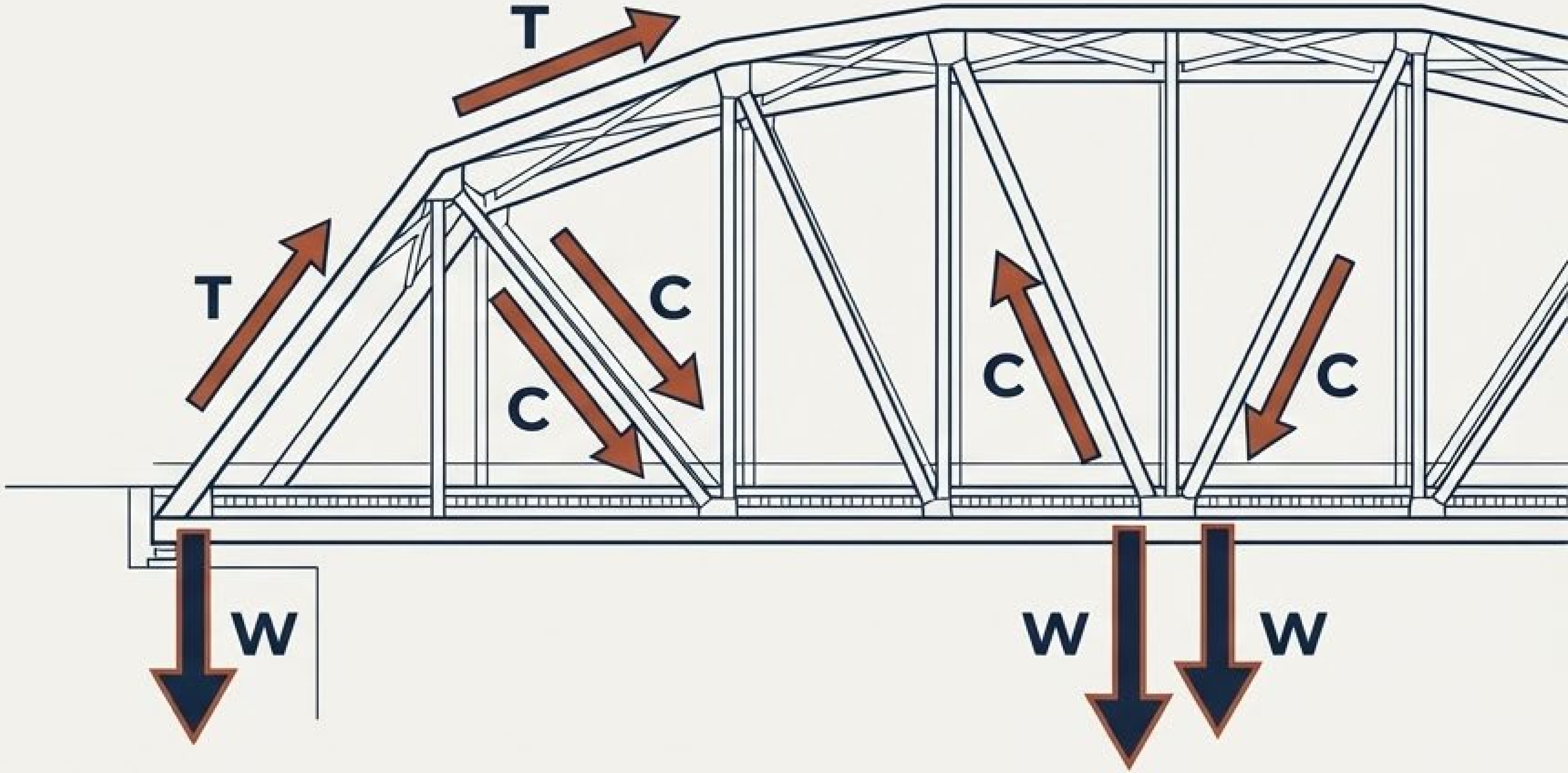


The Bridge Between Engineering and Medicine

An introduction to Mechanics, Biomechanics, and the foundational journey into Statics.

Prepared by Dr. Saif Mohammed Jawad

General Mechanics studies how forces impact predictable, non-living structures.



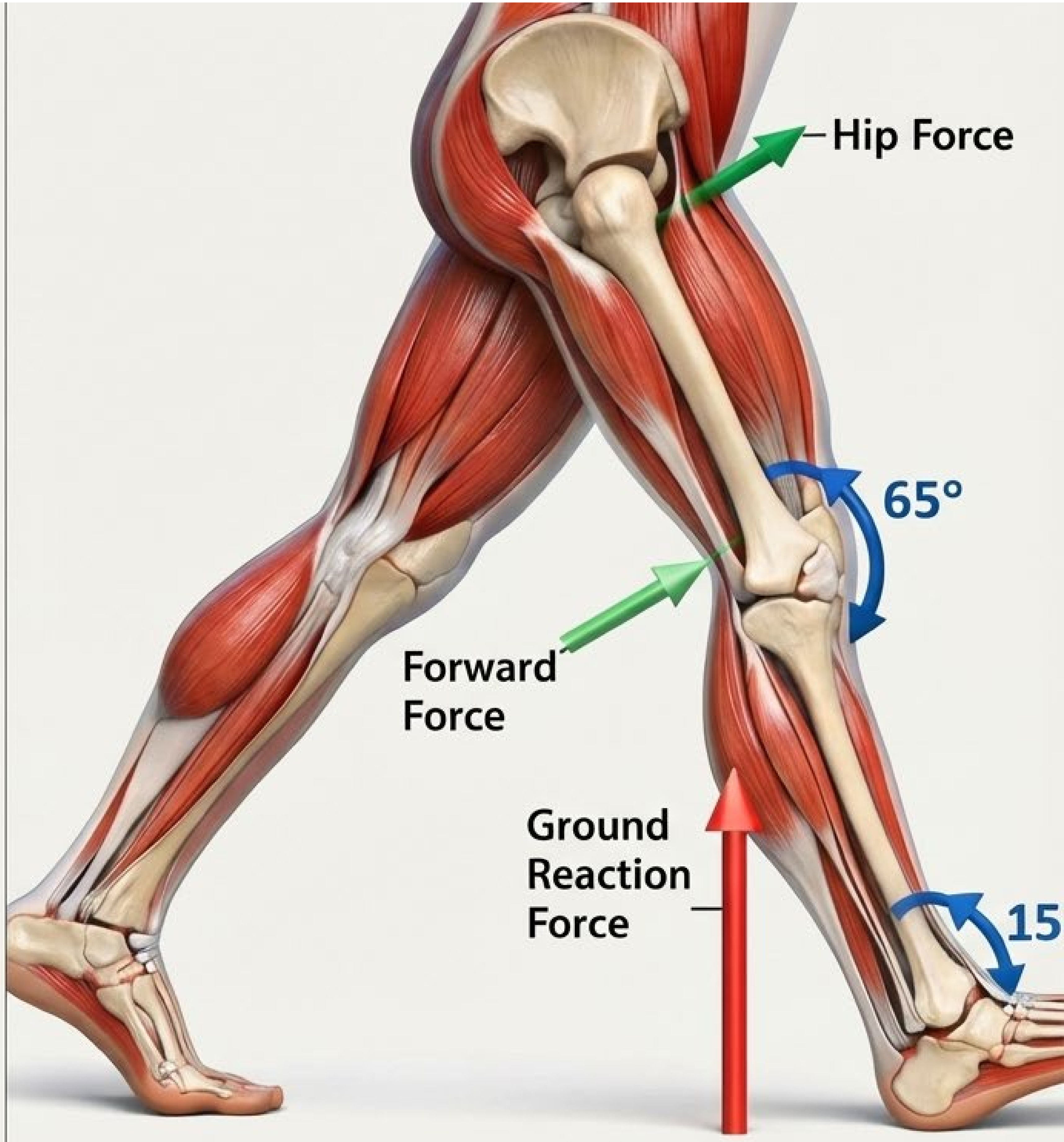
- **Definition:**
  The physical science studying objects at rest or in motion under the influence of forces.
- **Divisions:**
  Statics (equilibrium and rest) and Dynamics (motion).
- **Material Focus:**
  Deals with solid, homogeneous, and highly predictable materials like steel and concrete.

Biomechanics applies fundamental physics to dynamic biological systems.

Definition:
Using mechanics to study the structure and function of living systems, specifically the human body.

Scope:
Examines how bones, muscles, ligaments, and tendons work in unison to produce movement.

Target:
Understanding how the body responds to internal forces (muscle contractions) and external forces (gravity, ground reaction).



Biological systems introduce complex variables that traditional mechanics cannot predict.

	Traditional Mechanics	Biomechanics
Material	Inanimate (metals, polymers)	Living Tissues (bones, muscles)
Properties	Homogeneous and constant	Heterogeneous and variable (changes with age and health)
Response	Passive (responds only to external forces)	Active (capable of muscle contraction and growth)
Geometry	Regular shapes (cubes, cylinders, beams)	Complex, irregular shapes (joint surfaces, bones)

Mastering biomechanical forces is non-negotiable for clinical success in Prosthetics and Orthotics.

Load Distribution

Distributing pressure between the prosthesis and residual limb (stump) to prevent ulcers.

Alignment

Adjusting angles to guarantee patient stability and balance forces during standing and walking.

Gait Analysis

Studying movement forces to reduce patient effort and improve walking efficiency.

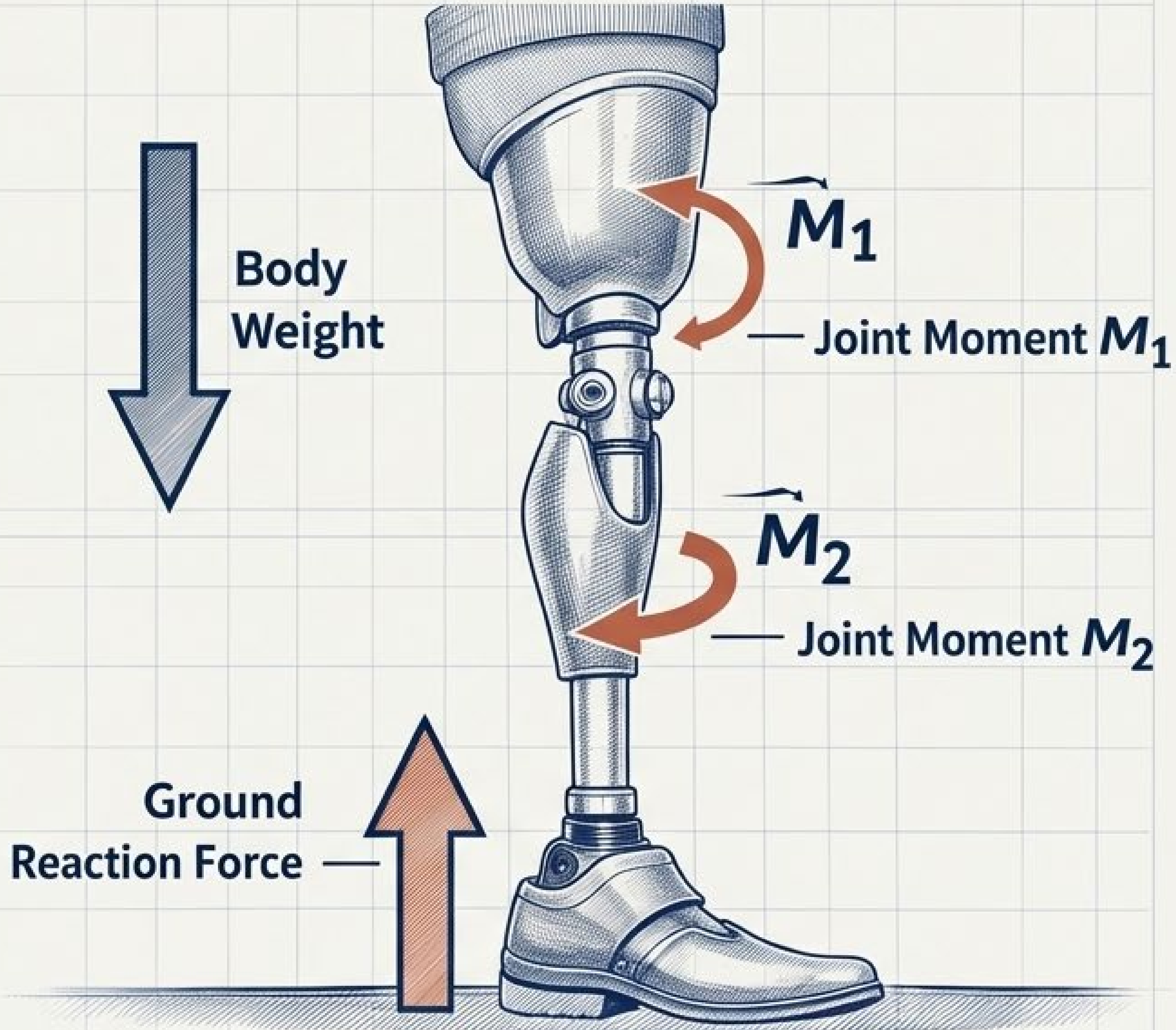
Mechanical Efficiency

Designing orthoses that properly support weak joints and prevent future structural deformities.

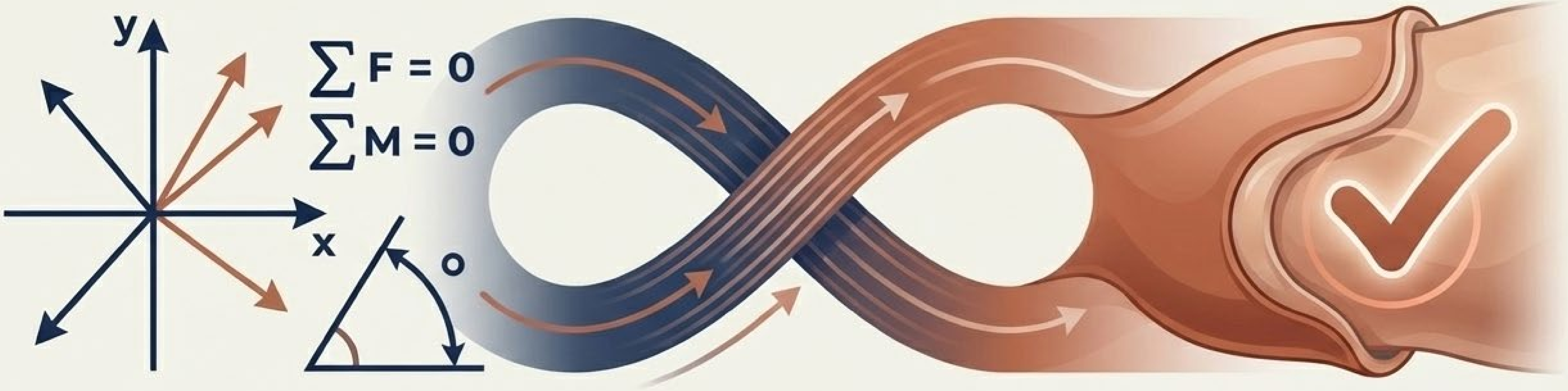
Statics provides the specific mathematical foundation for patient stability and device equilibrium.

Definition: The precise analysis of forces and moments acting on objects in complete rest or equilibrium.

Core Topics: Mastery of Vectors, Centers of Gravity, and the equilibrium of particles and rigid bodies.



Mathematical principles translate directly into patient comfort and tissue safety.



The Practical Connection

Understanding exactly how a patient maintains balance and how a prosthesis distributes forces to prevent falls.

The Ultimate Benefit

Utilizing static equations to calculate exact pressures inside the prosthetic socket, ensuring absolute patient comfort and preventing tissue injury.

The journey to clinical mastery begins with understanding equilibrium.

The Synthesis

Mechanics provides the mathematical tools; Biomechanics applies them to understand human movement.

The Prerequisite

Before we can safely manage complex, dynamic motion, we must master static equilibrium to guarantee patient safety.

The Goal

Ultimately, achieving stability and balance is the foundation of effective patient care and prosthetic design.

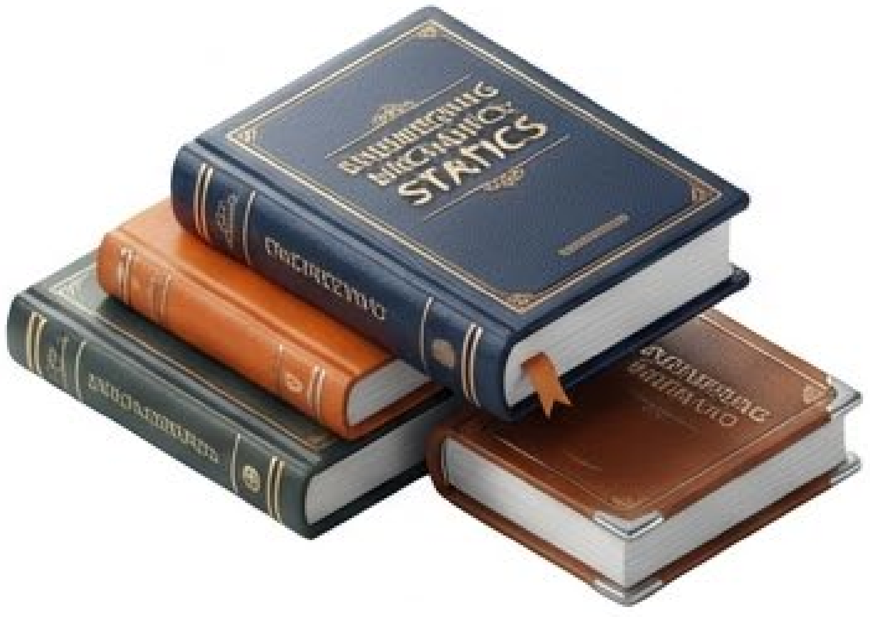
Ready to begin the Statics journey?

Engineering Mechanics: Statics

Chapter 1: General Principles & Fundamental Concepts

The Road Map to Statics

- Introduction To Statics
- Force Vectors & Force Systems
- Equilibrium & Structure Analysis
- Friction
- Center of Mass & Centroids
- Moment of Inertia & Virtual Work



Textbooks & References:

Textbooks | Statics

- R.C. Hibbeler (14th Ed.) - Primary
- F.P. Beer - Reference
- J.L. Meriam - Reference

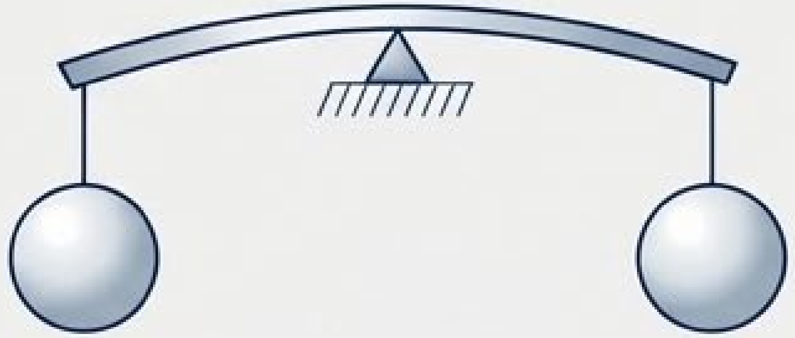
What is Mechanics?

“Mechanics is a branch of the physical sciences concerned with the state of rest or motion of bodies subjected to the action of forces.”

The Branches of Mechanics

Mechanics

- Rigid-Body Mechanics
- Deformable-Body Mechanics
- Fluid Mechanics

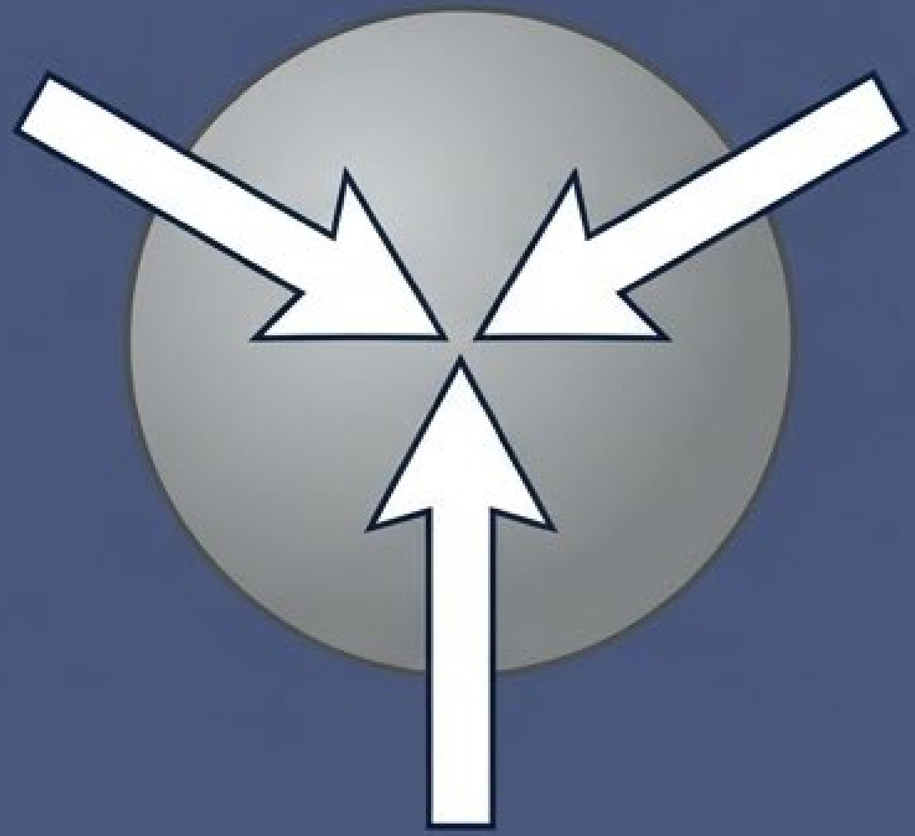


Our focus is on Rigid-Body Mechanics, which assumes the body's shape does not change under load.

Statics vs. Dynamics

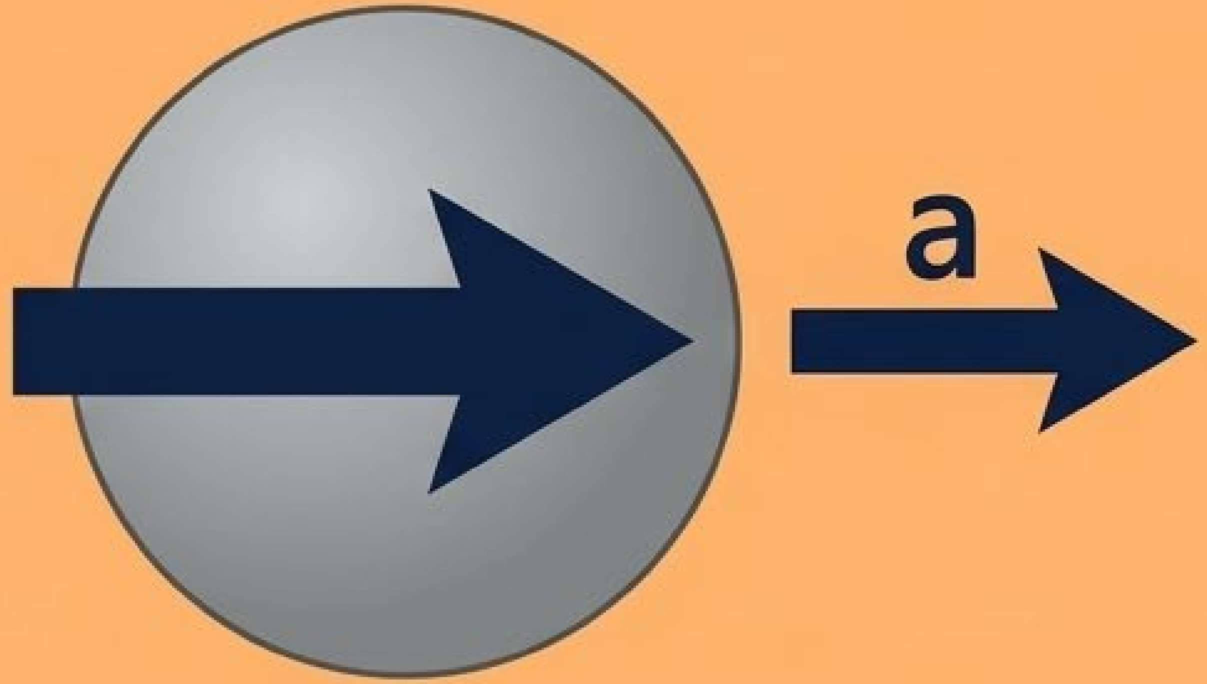
Statics (The Special Case: $a = 0$)

- Deals with the equilibrium of bodies.
- Bodies are either at rest or moving with a constant velocity.
- Pioneers: Archimedes, Stevinus.



Dynamics

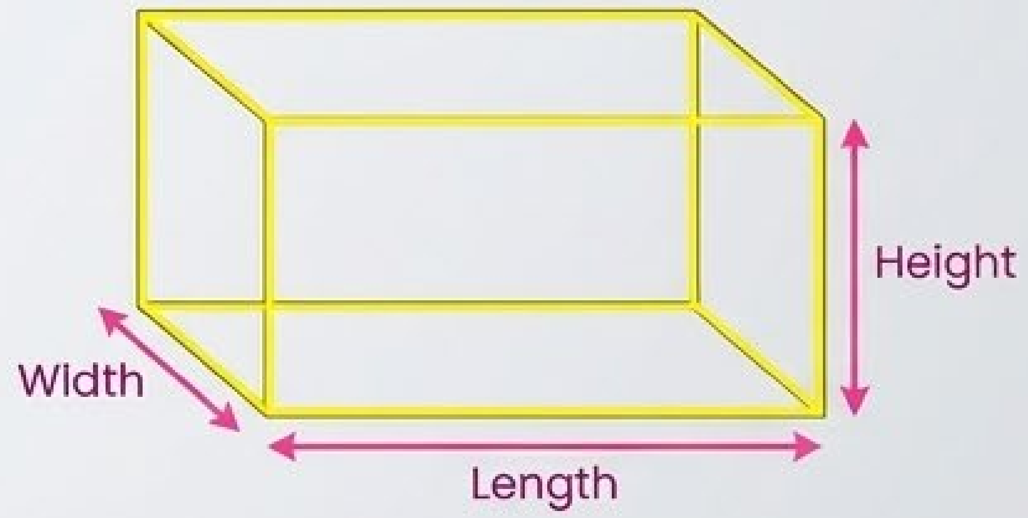
- Concerned with the accelerated motion of bodies.
- Pioneers: Galileo, Newton.



The Fundamental Quantities

These four basic quantities are the building blocks used throughout all mechanics.

Length



Time

Mass

Force

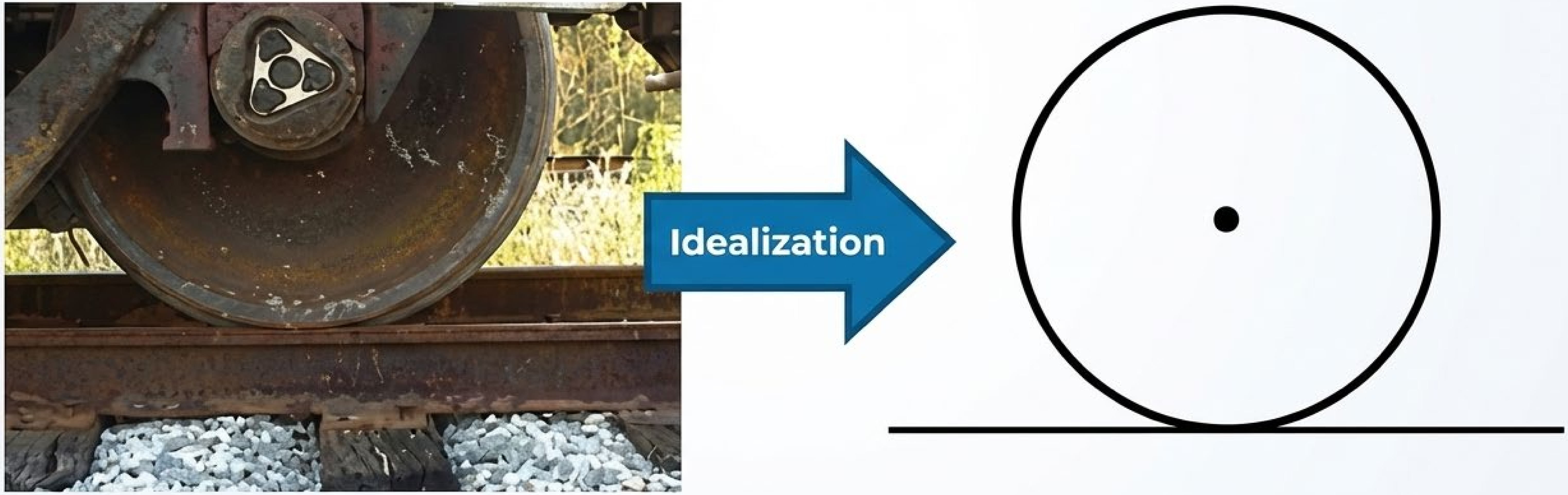
Units of Measurement

Name	Length	Time	Mass	Force
International System of Units (SI)	meter (m)	second (s)	kilogram (kg)	newton (N)
U.S. Customary (FPS)	foot (ft)	second (s)	slug	pound (lb)

Simplifying Reality: Engineering Idealizations

Models or idealizations are used in mechanics to simplify the application of theory without losing accuracy.

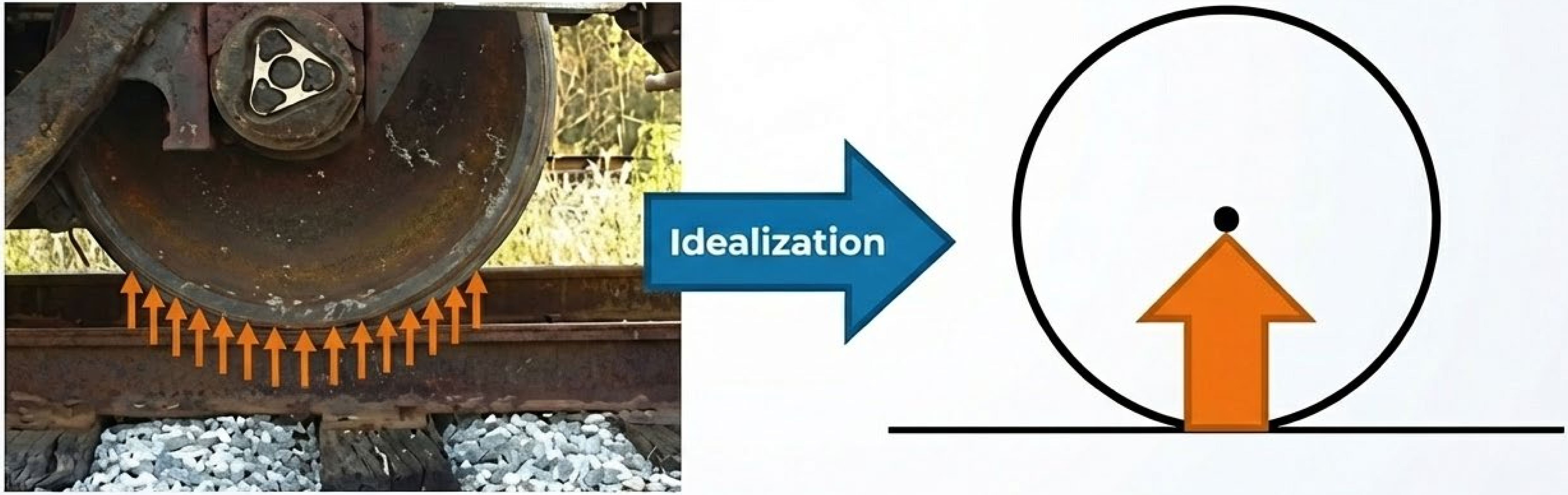
Reality | **Engineering Model**



Idealization: Concentrated Force

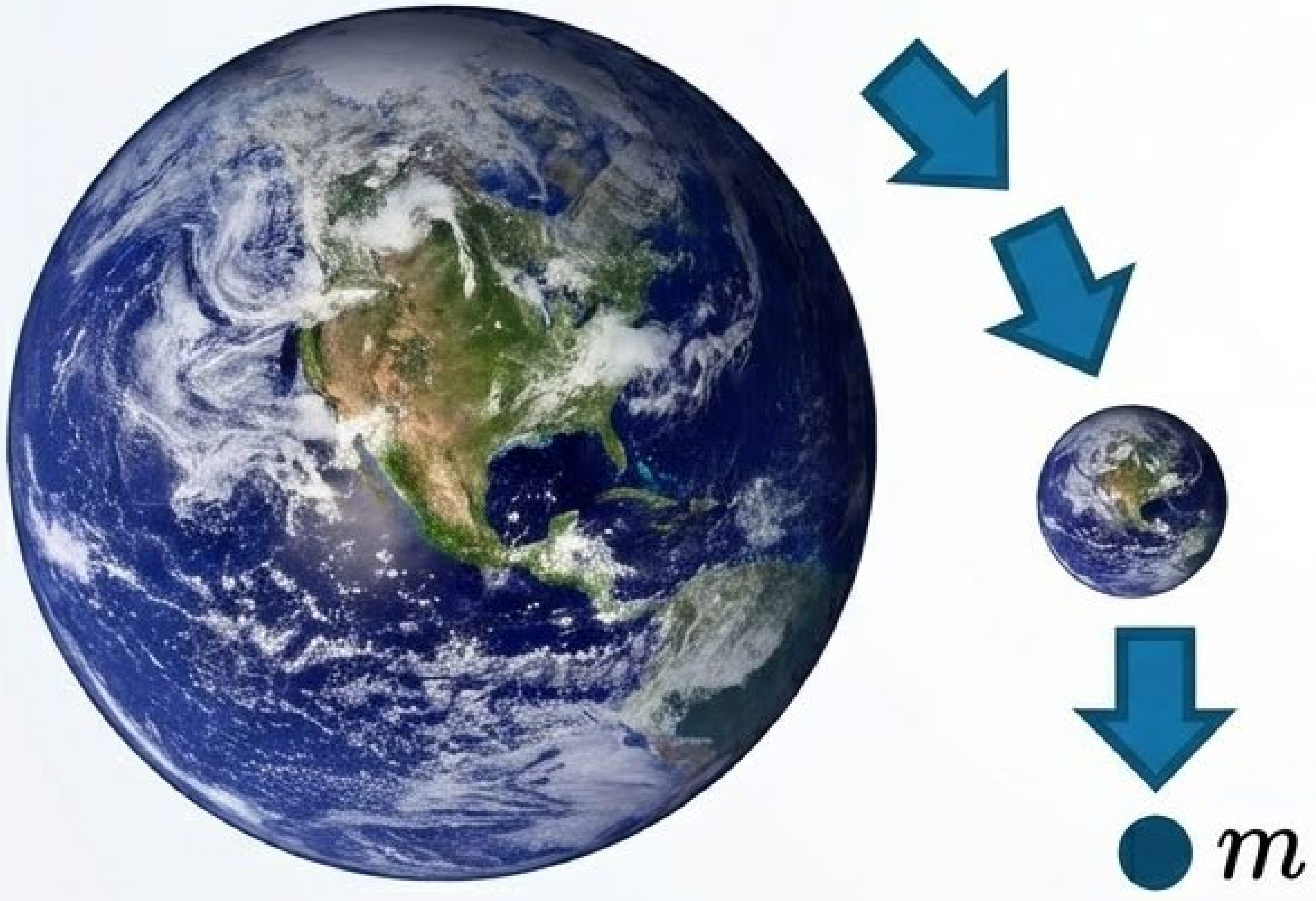
A concentrated force represents the effect of a loading which is assumed to act at a single point on a body.

Reality | **Concentrated Force (Idealization)**

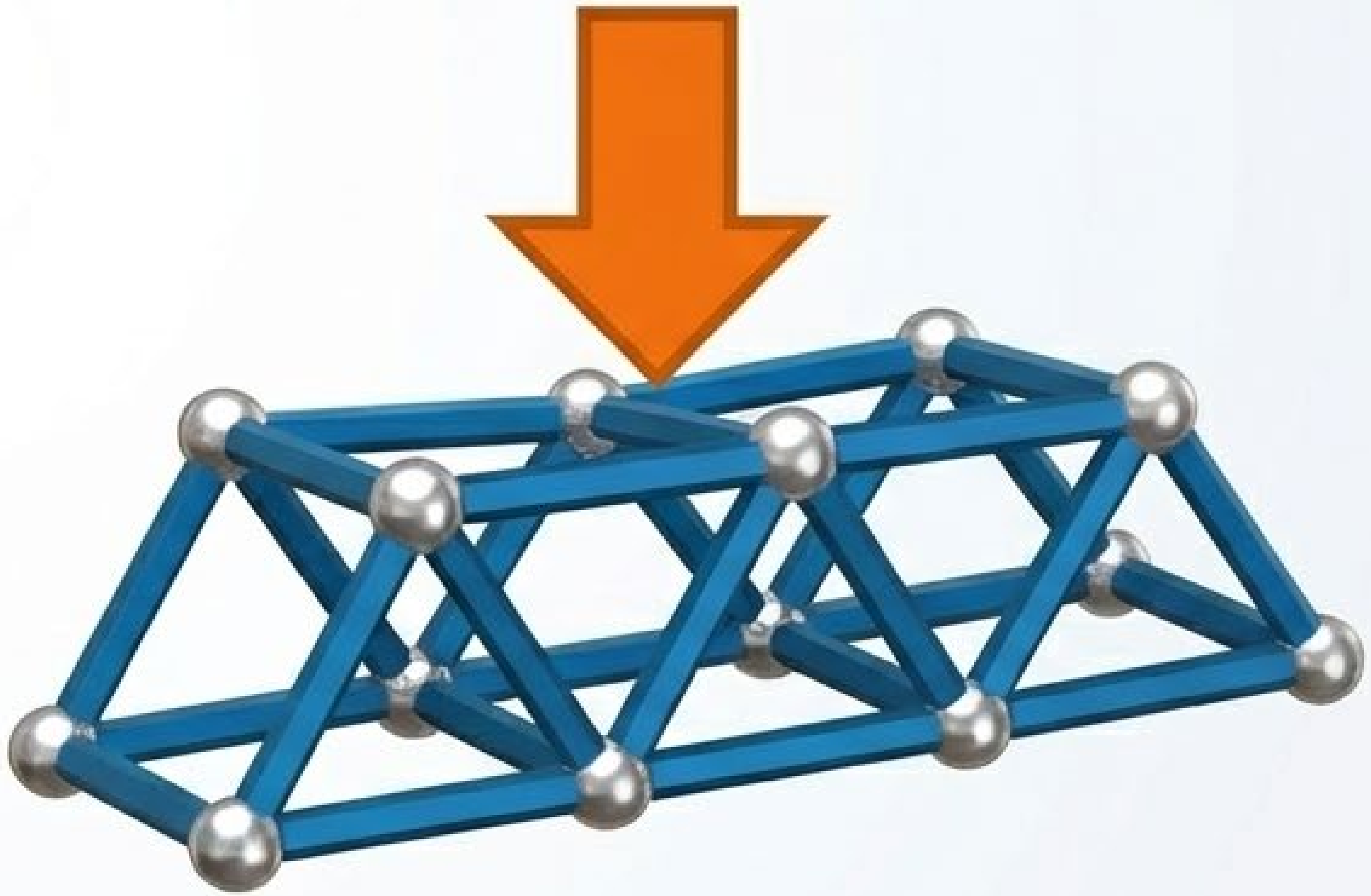


Idealizations: Particles & Rigid Bodies

Particle: Has a mass, but its size can be neglected in calculations.



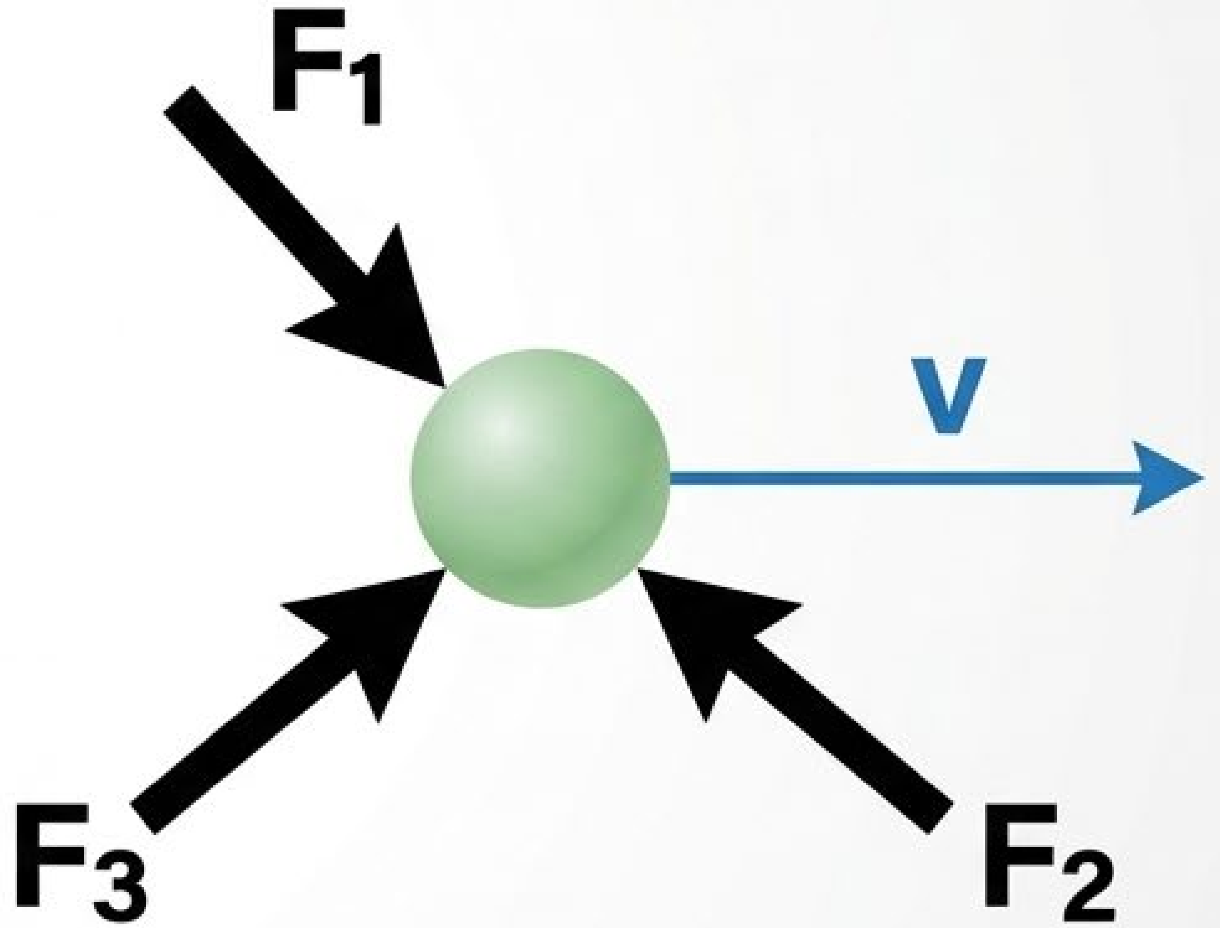
Rigid Body: A combination of a large number of particles where all particles remain at a fixed distance from one another, before and after applying a load.



NotebookLM

Newton's First Law: Inertia & Equilibrium

A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is not subjected to an unbalanced force.

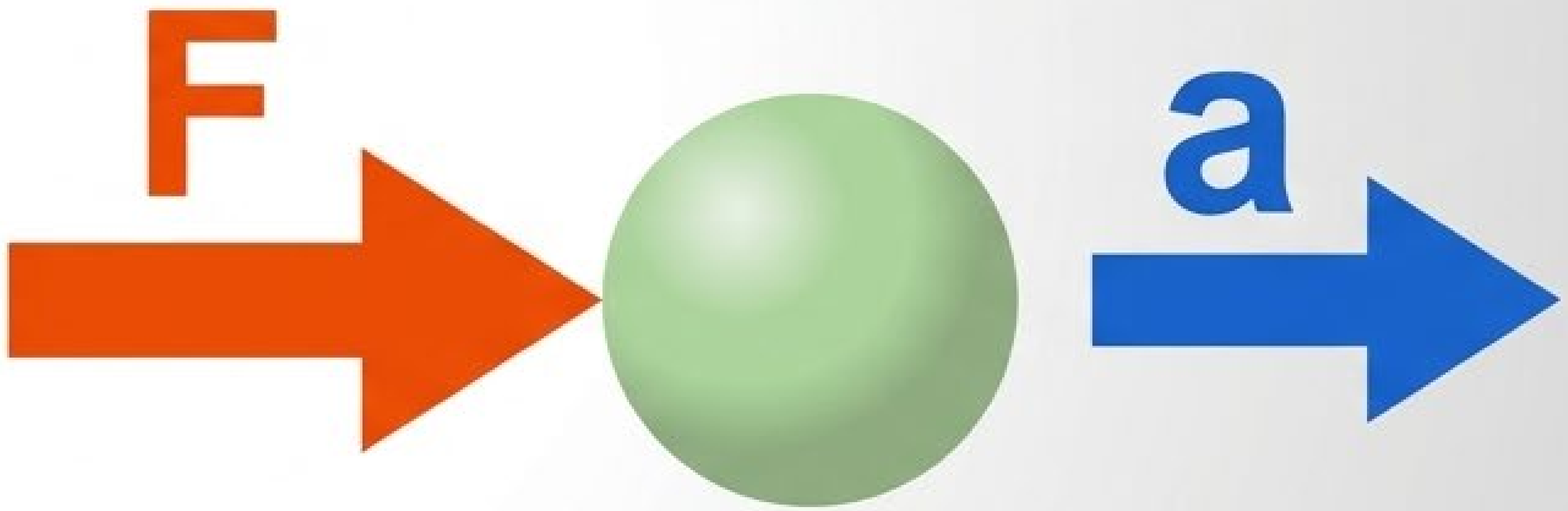


Equilibrium

Newton’s Second Law: Accelerated Motion

A particle acted upon by an unbalanced force (**F**) experiences an acceleration (**a**) that has the same direction as the force and a magnitude **directly proportional to** **to** the force.

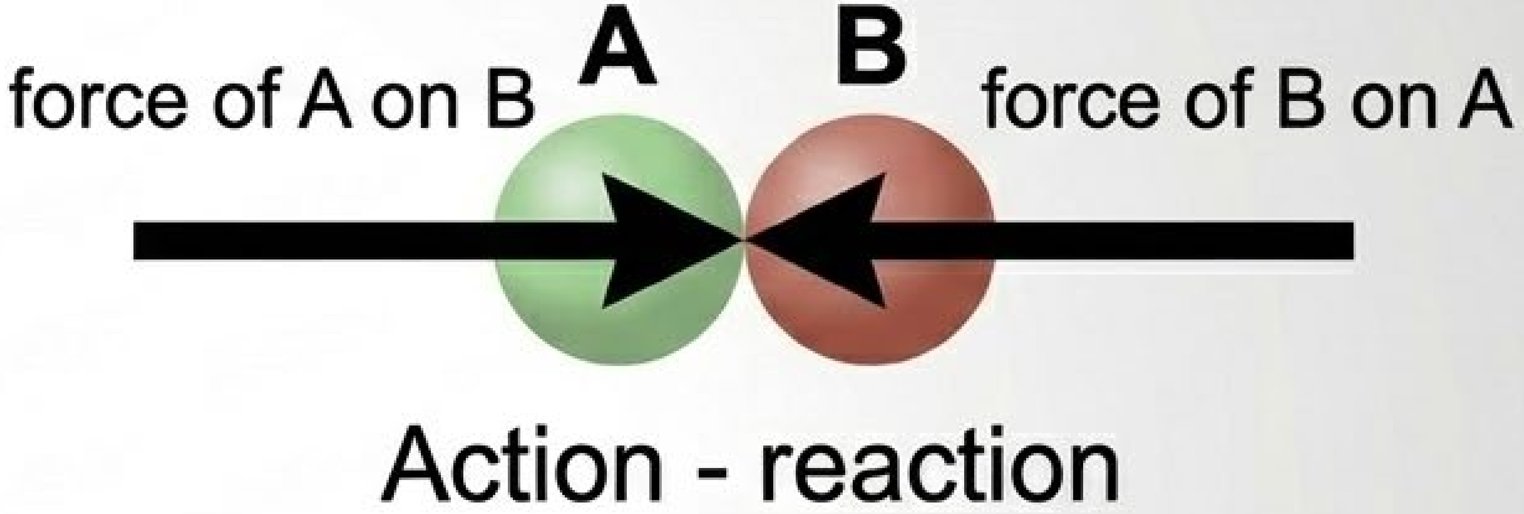
$$F = ma$$



Accelerated motion

Newton’s Third Law: Action and Reaction

The mutual forces of action and reaction between two particles are equal, opposite, and collinear.



Newton's Law of Gravitational Attraction

This law governs the gravitational attraction between any two particles in the universe.

m_1, m2 = mass of particles

$$F = G(m_1 m_2)/r_2$$

F = force of gravitation

G = universal constant

r = distance between particles

From Gravity to Weight

When a particle is located at or near the surface of the earth, the complex universal gravitation equation simplifies dramatically.

$$W = G(mM_e)/r2$$

Letting $g = GM_e/r2$ yields

$$W = mg$$

Lecture 2

Force Vectors

CHAPTER OBJECTIVES

- To show how to add forces and resolve them into components using the Parallelogram Law.
- To express force and position in Cartesian vector form and explain how to determine the vector's magnitude and direction.
- To introduce the dot product in order to use it to find the angle between two vectors or the projection of one vector onto another.

2.1 Scalars and Vectors

Many physical quantities in engineering mechanics are measured using either scalars or vectors.

Scalar. A ***scalar*** is any positive or negative physical quantity that can be completely specified by its *magnitude*. Examples of scalar quantities include length, mass, and time.

Vector. A ***vector*** is any physical quantity that requires both a *magnitude* and a *direction* for its complete description. Examples of vectors encountered in statics are force, position, and moment. A vector is shown graphically by an arrow. The length of the arrow represents the *magnitude* of the vector, and the angle θ between the vector and a fixed axis defines the *direction of its line of action*. The head or tip of the arrow indicates the *sense of direction* of the vector, Fig. 2–1.

In print, vector quantities are represented by boldface letters such as **A**, and the magnitude of a vector is italicized, *A*. For handwritten work, it is often convenient to denote a vector quantity by simply drawing an arrow above it, $\vec{A}$.

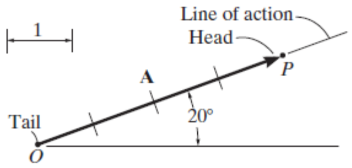
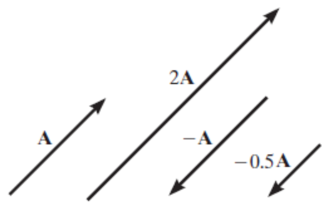


Fig. 2–1

2.2 Vector Operations



Scalar multiplication and division

Fig. 2–2

Multiplication and Division of a Vector by a Scalar. If a vector is multiplied by a positive scalar, its magnitude is increased by that amount. Multiplying by a negative scalar will also change the directional sense of the vector. Graphic examples of these operations are shown in Fig. 2–2.

Vector Addition. When adding two vectors together it is important to account for both their magnitudes and their directions. To do this we must use the ***parallelogram law of addition***. To illustrate, the two ***component vectors*** **A** and **B** in Fig. 2–3*a* are added to form a ***resultant vector*** **R** = **A** + **B** using the following procedure:

- First join the tails of the components at a point to make them concurrent, Fig. 2–3*b*.
- From the head of **B**, draw a line parallel to **A**. Draw another line from the head of **A** that is parallel to **B**. These two lines intersect at point *P* to form the adjacent sides of a parallelogram.
- The diagonal of this parallelogram that extends to *P* forms **R**, which then represents the resultant vector **R** = **A** + **B**, Fig. 2–3*c*.

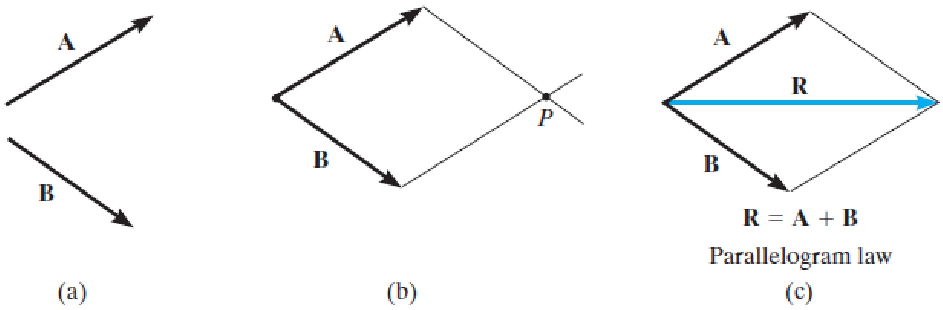


Fig. 2–3

We can also add **B** to **A**, Fig. 2–4*a*, using the ***triangle rule***, which is a special case of the parallelogram law, whereby vector **B** is added to vector **A** in a "head-to-tail" fashion, i.e., by connecting the head of **A** to the tail of **B**, Fig. 2–4*b*. The resultant **R** extends from the tail of **A** to the head of **B**. In a similar manner, **R** can also be obtained by adding **A** to **B**, Fig. 2–4*c*. By comparison, it is seen that vector addition is commutative; in other words, the vectors can be added in either order, i.e., $\mathbf{R} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

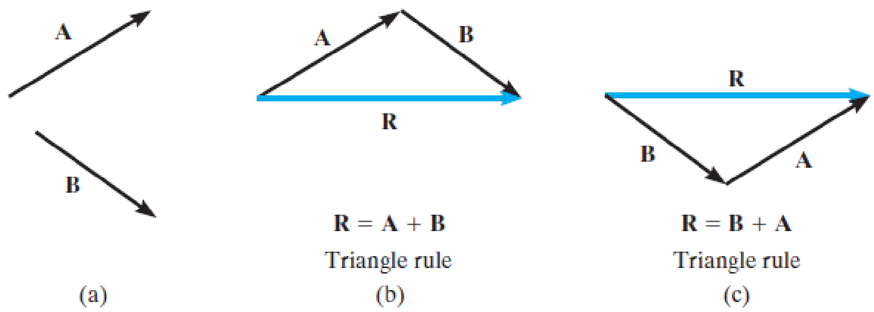
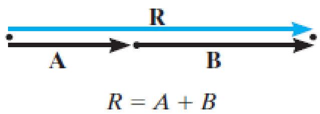


Fig. 2–4

As a special case, if the two vectors **A** and **B** are ***collinear***, i.e., both have the same line of action, the parallelogram law reduces to an *algebraic* or *scalar addition* $R = A + B$, as shown in Fig. 2–5.



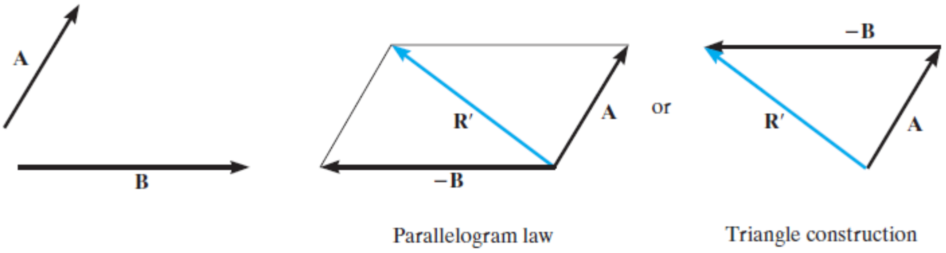
Addition of collinear vectors

Fig. 2–5

Vector Subtraction. The resultant of the *difference* between two vectors **A** and **B** of the same type may be expressed as

$$\mathbf{R}' = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

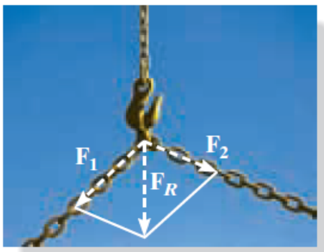
This vector sum is shown graphically in Fig. 2–6. Subtraction is therefore defined as a special case of addition, so the rules of vector addition also apply to vector subtraction.



Vector subtraction

Fig. 2–6

2.3 Vector Addition of Forces



The parallelogram law must be used to determine the resultant of the two forces acting on the hook. (© Russell C. Hibbeler)

Experimental evidence has shown that a force is a vector quantity since it has a specified magnitude, direction, and sense and it adds according to the parallelogram law. Two common problems in statics involve either finding the resultant force, knowing its components, or resolving a known force into two components. We will now describe how each of these problems is solved using the parallelogram law.

Finding a Resultant Force. The two component forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the pin in Fig. 2–7*a* can be added together to form the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$, as shown in Fig. 2–7*b*. From this construction, or using the triangle rule, Fig. 2–7*c*, we can apply the law of cosines or the law of sines to the triangle in order to obtain the magnitude of the resultant force and its direction.

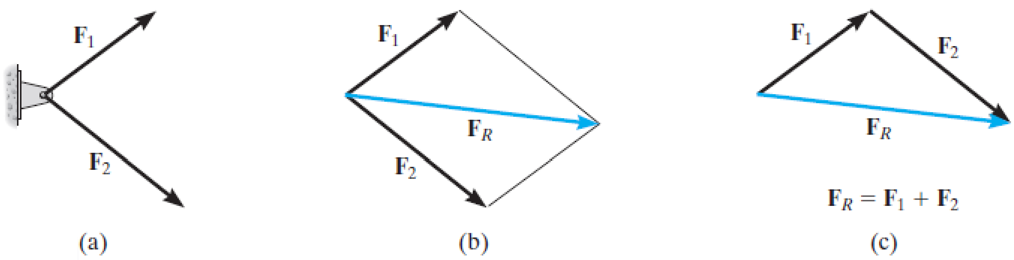


Fig. 2–7

Finding the Components of a Force. Sometimes it is necessary to resolve a force into two *components* in order to study its pulling or pushing effect in two specific directions. For example, in Fig. 2–8*a*, **F** is to be resolved into two components along the two members, defined by the *u* and *v* axes. In order to determine the magnitude of each component, a parallelogram is constructed first, by drawing lines starting from the tip of **F**, one line parallel to *u*, and the other line parallel to *v*. These lines then intersect with the *v* and *u* axes, forming a parallelogram. The force components $\mathbf{F}_u$ and $\mathbf{F}_v$ are then established by simply joining the tail of **F** to the intersection points on the *u* and *v* axes, Fig. 2–8*b*. This parallelogram can then be reduced to a triangle, which represents the triangle rule, Fig. 2–8*c*. From this, the law of sines can then be applied to determine the unknown magnitudes of the components.

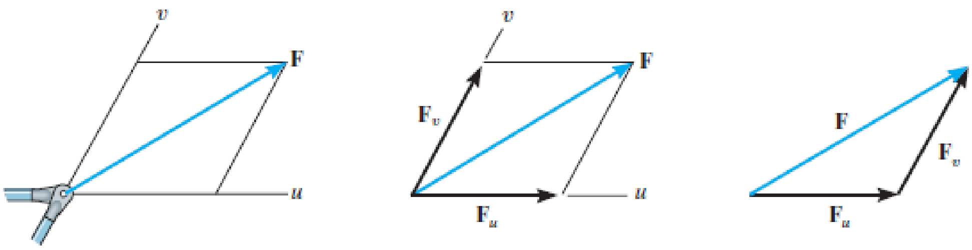
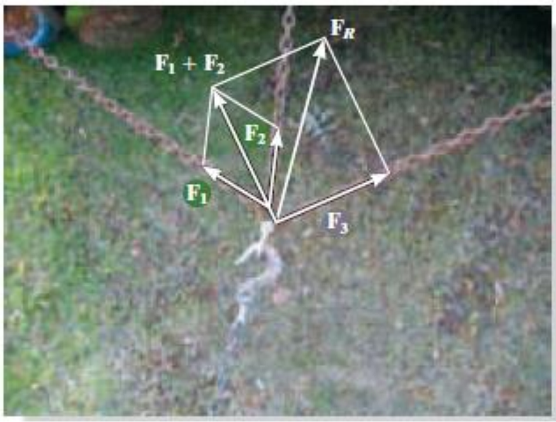


Fig. 2–8

Addition of Several Forces. If more than two forces are to be added, successive applications of the parallelogram law can be carried out in order to obtain the resultant force. For example, if three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act at a point *O*, Fig. 2–9, the resultant of any two of the forces is found, say, $\mathbf{F}_1 + \mathbf{F}_2$—and then this resultant is added to the third force, yielding the resultant of all three forces; i.e., $\mathbf{F}_R = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$. Using the parallelogram law to add more than two forces, as shown here, often requires extensive geometric and trigonometric calculation to determine the numerical values for the magnitude and direction of the resultant. Instead, problems of this type are easily solved by using the "rectangular-component method," which is explained in Sec. 2.4.



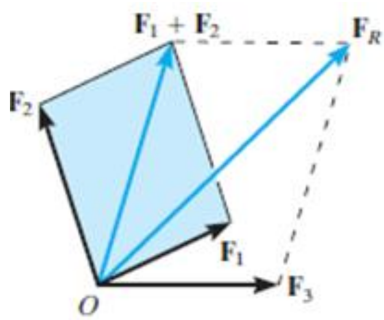


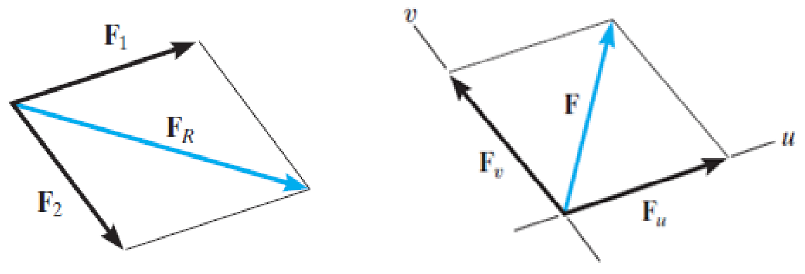
Fig. 2–9

Procedure for Analysis

Problems that involve the addition of two forces can be solved as follows:

1. Parallelogram Law:

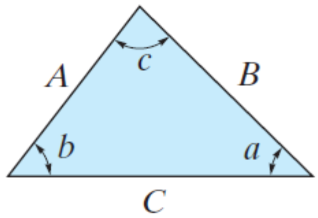
- Two "component" forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. below, add according to the parallelogram law, yielding a *resultant* force $\mathbf{F}_R$ that forms the diagonal of the parallelogram.
- If a force **F** is to be resolved into *components* along two axes *u* and v, Fig. 2–10*b*, then start at the head of force **F** and constructn lines parallel to the axes, thereby forming the parallelogram. The sides of the parallelogram represent the components, **F***u* and **F**v.
- Label all the known and unknown force magnitudes and the angles on the sketch and identify the two unknowns as the magnitude and direction of $\mathbf{F}_R$, or the magnitudes of its components.



(a) (b)

2. Trigonometry

- Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components.
- From this triangle, the magnitude of the resultant force can be determined using the law of cosines, and its direction is determined from the law of sines. The magnitudes of two force components are determined from the law of sines. The formulas are given in Fig. below.



Cosine law:

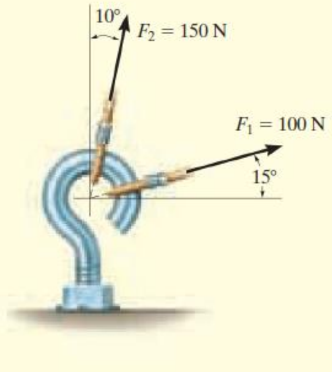
$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

Sine law:

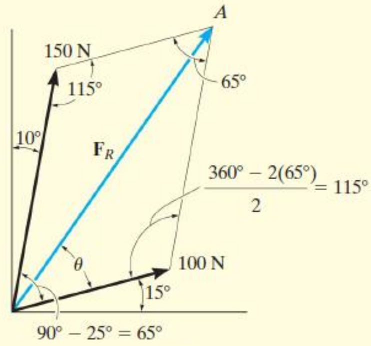
$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

EXAMPLE 2.1

The screw eye in Fig. 2–11*a* is subjected to two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.



(a)



(b)

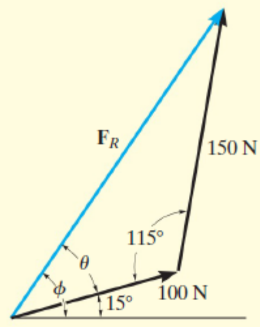
SOLUTION

Parallelogram Law. The parallelogram is formed by drawing a line from the head of $\mathbf{F}_1$ that is parallel to $\mathbf{F}_2$, and another line from the head of $\mathbf{F}_2$ that is parallel to $\mathbf{F}_1$. The resultant force $\mathbf{F}_R$ extends to where these lines intersect at point *A*, Fig. 2–11*b*. The two unknowns are the magnitude of $\mathbf{F}_R$ and the angle θ (theta).

Trigonometry. From the parallelogram, the vector triangle is constructed, Fig. 2–11*c*. Using the law of cosines

$$F_R = \sqrt{(100\text{ N})^2 + (150\text{ N})^2 - 2(100\text{ N})(150\text{ N})\cos 115^\circ}$$
$$= \sqrt{10\,000 + 22\,500 - 30\,000(-0.4226)} = 212.6\text{ N}$$
$$= 213\text{ N} \qquad \textit{Ans.}$$

Applying the law of sines to determine θ,



(c)

Fig. 2–11

$$\frac{150\text{ N}}{\sin\theta} = \frac{212.6\text{ N}}{\sin 115^\circ} \qquad \sin\theta = \frac{150\text{ N}}{212.6\text{ N}}(\sin 115^\circ)$$
$$\theta = 39.8^\circ$$

Thus, the direction ϕ (phi) of $\mathbf{F}_R$, measured from the horizontal, is

$$\phi = 39.8^\circ + 15.0^\circ = 54.8^\circ \qquad \textit{Ans.}$$

NOTE: The results seem reasonable, since Fig. 2–11*b* shows $\mathbf{F}_R$ to have a magnitude larger than its components and a direction that is between them.

EXAMPLE 2.2

Resolve the horizontal 600-lb force in Fig. 2–12*a* into components acting along the *u* and *v* axes and determine the magnitudes of these components.

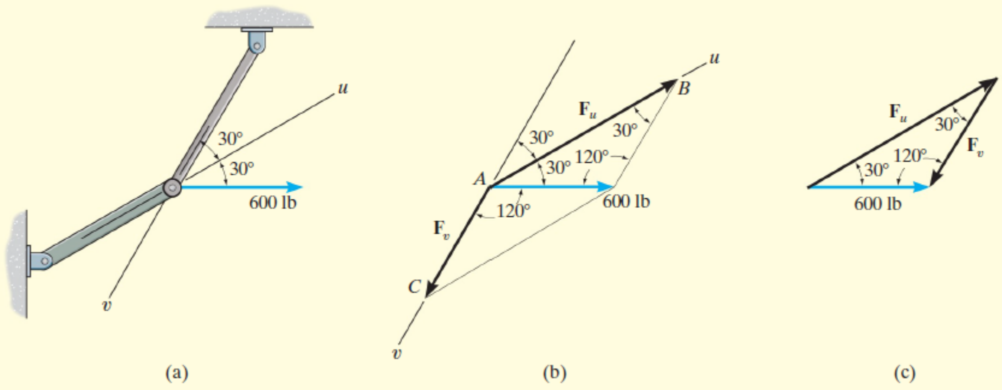


Fig. 2–12

SOLUTION

The parallelogram is constructed by extending a line from the *head* of the 600-lb force parallel to the *v* axis until it intersects the *u* axis at point *B*, Fig. 2–12*b*. The arrow from *A* to *B* represents $\mathbf{F}_u$. Similarly, the line extended from the head of the 600-lb force drawn parallel to the *u* axis intersects the *v* axis at point *C*, which gives $\mathbf{F}_v$.

The vector addition using the triangle rule is shown in Fig. 2–12*c*. The two unknowns are the magnitudes of $\mathbf{F}_u$ and $\mathbf{F}_v$. Applying the law of sines,

$$\frac{F_u}{\sin 120^\circ} = \frac{600 \text{ lb}}{\sin 30^\circ}$$

$$F_u = 1039 \text{ lb} \qquad \textit{Ans.}$$

$$\frac{F_v}{\sin 30^\circ} = \frac{600 \text{ lb}}{\sin 30^\circ}$$

$$F_v = 600 \text{ lb} \qquad \textit{Ans.}$$

NOTE: The result for F_u shows that sometimes a component can have a greater magnitude than the resultant.

EXAMPLE 2.3

Determine the magnitude of the component force **F** in Fig. 2–13*a* and the magnitude of the resultant force $\mathbf{F}_R$ if $\mathbf{F}_R$ is directed along the positive *y* axis.

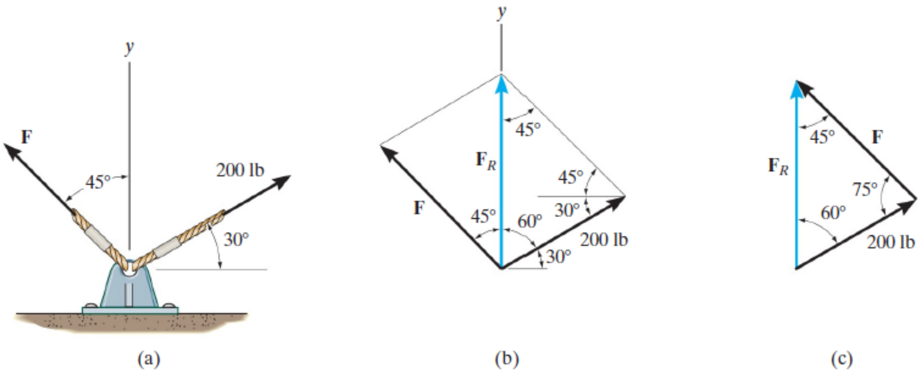


Fig. 2–13

SOLUTION

The parallelogram law of addition is shown in Fig. 2–13*b*, and the triangle rule is shown in Fig. 2–13*c*. The magnitudes of $\mathbf{F}_R$ and $\mathbf{F}$ are the two unknowns. They can be determined by applying the law of sines.

$$\frac{F}{\sin 60^\circ} = \frac{200 \text{ lb}}{\sin 45^\circ}$$

$$F = 245 \text{ lb} \qquad \textit{Ans.}$$

$$\frac{F_R}{\sin 75^\circ} = \frac{200 \text{ lb}}{\sin 45^\circ}$$

$$F_R = 273 \text{ lb} \qquad \textit{Ans.}$$

EXAMPLE 2.4

It is required that the resultant force acting on the eyebolt in Fig. 2–14*a* be directed along the positive *x* axis and that $\mathbf{F}_2$ have a *minimum* magnitude. Determine this magnitude, the angle θ, and the corresponding resultant force.

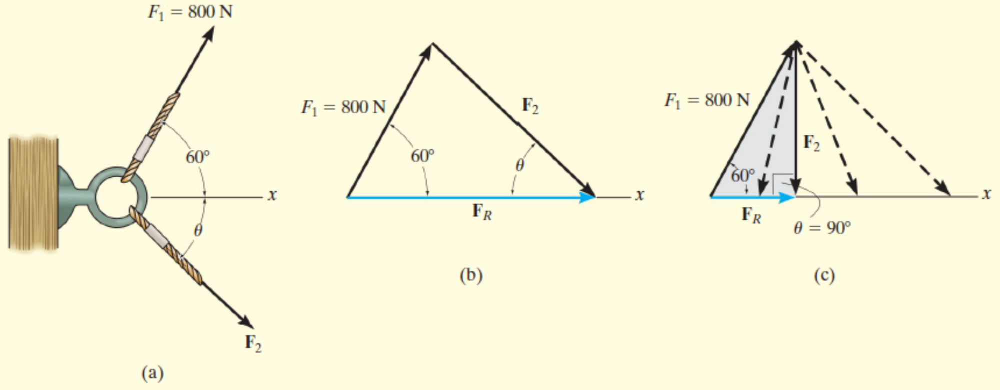


Fig. 2–14

SOLUTION

The triangle rule for $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ is shown in Fig. 2–14*b*. Since the magnitudes (lengths) of $\mathbf{F}_R$ and $\mathbf{F}_2$ are not specified, then $\mathbf{F}_2$ can actually be any vector that has its head touching the line of action of $\mathbf{F}_R$, Fig. 2–14*c*. However, as shown, the magnitude of $\mathbf{F}_2$ is a *minimum* or the shortest length when its line of action is *perpendicular* to the line of action of $\mathbf{F}_R$, that is, when

$$\theta = 90^\circ \qquad \textit{Ans.}$$

Since the vector addition now forms the shaded right triangle, the two unknown magnitudes can be obtained by trigonometry.

$$F_R = (800\text{ N})\cos 60^\circ = 400\text{ N} \qquad \textit{Ans.}$$

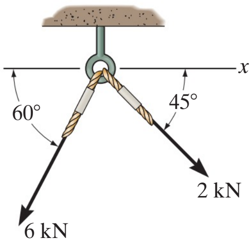
$$F_2 = (800\text{ N})\sin 60^\circ = 693\text{ N} \qquad \textit{Ans.}$$

Note: It is strongly suggested that you test yourself on the solutions to these examples, by covering them over and then trying to draw the parallelogram law, and thinking about how the sine and cosine laws are used to determine the unknowns. Then before solving any of the problems, try to solve the the Fundamental Problems given on the next pages. The solutions and answers to these are given in the back of the book. Doing this throughout the book will help immensely in developing your problem-solving skills.

FUNDAMENTAL PROBLEMS

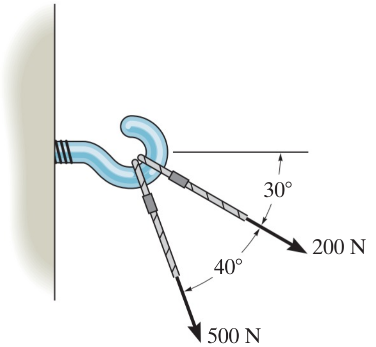
Partial solutions and answers to all Fundamental Problems are given in the back of the book.

F2–1. Determine the magnitude of the resultant force acting on the screw eye and its direction measured clockwise from the x axis.



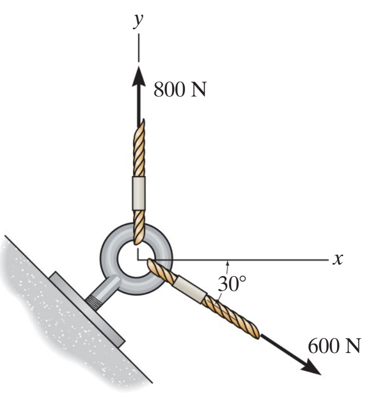
Prob. F2–1

F2–2. Two forces act on the hook. Determine the magnitude of the resultant force.



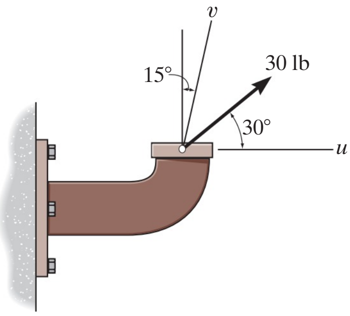
Prob. F2–2

F2–3. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



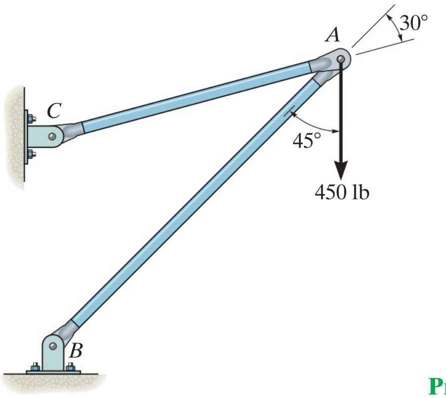
Prob. F2–3

F2–4. Resolve the 30-lb force into components along the u and v axes, and determine the magnitude of each of these components.



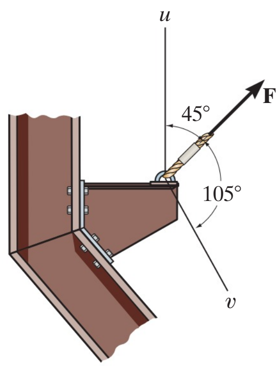
Prob. F2–4

F2–5. The force $F = 450$ lb acts on the frame. Resolve this force into components acting along members AB and AC, and determine the magnitude of each component.



Prob. F2–5

F2–6. If force **F** is to have a component along the u axis of $F_u = 6$ kN, determine the magnitude of **F** and the magnitude of its component $\mathbf{F}_v$ along the v axis.



Prob. F2–6

2.4 Addition of a System of Coplanar Forces

When a force is resolved into two components along the x and y axes, the components are then called ***rectangular components***. For analytical work we can represent these components in one of two ways, using either scalar or Cartesian vector notation.

Scalar Notation. The rectangular components of force **F** shown in Fig. 2–15*a* are found using the parallelogram law, so that $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$. Because these components form a right triangle, they can be determined from

$$F_x = F\cos\theta \quad \text{and} \quad F_y = F\sin\theta$$

Instead of using the angle θ, however, the direction of **F** can also be defined using a small "slope" triangle, as in the example shown in Fig. 2–15*b*. Since this triangle and the larger shaded triangle are similar, the proportional length of the sides gives

$$\frac{F_x}{F} = \frac{a}{c}$$

or

$$F_x = F\left(\frac{a}{c}\right)$$

and

$$\frac{F_y}{F} = \frac{b}{c}$$

or

$$F_y = -F\left(\frac{b}{c}\right)$$

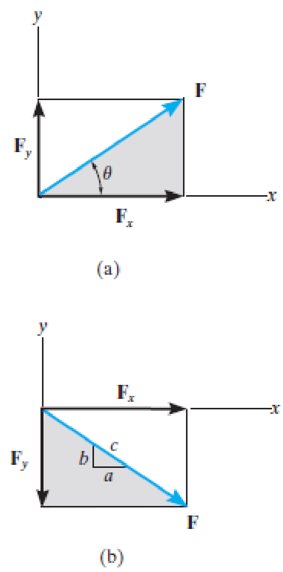


Fig. 2–15

Here the y component is a negative scalar since Fy is directed along the negative y axis. It is important to keep in mind that this positive and negative scalar notation is to be used only for computational purposes, not for graphical representations in figures. Throughout the book, the head of a vector arrow in any figure indicates the sense of the vector graphically; algebraic signs are not used for this purpose. Thus, the vectors in Figs. 2–15a and 2-15b are designated by using boldface (vector) notation.* Whenever italic symbols are written near vector arrows in figures, they indicate the magnitude of the vector, which is always a positive quantity

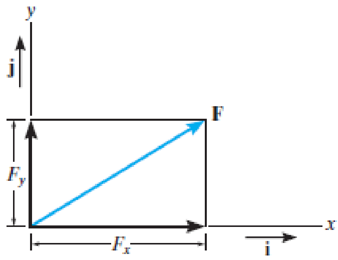


Fig. 2–16

Cartesian Vector Notation. It is also possible to represent the x and y components of a force in terms of Cartesian unit vectors **i** and **j**. They are called unit vectors because they have a dimensionless magnitude of 1, and so they can be used to designate the *directions* of the x and y axes, respectively, Fig. 2–16.*

Since the *magnitude* of each component of **F** is *always a positive quantity*, which is represented by the (positive) scalars F_x and F_y, then we can express **F** as a ***Cartesian vector***,

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j}$$

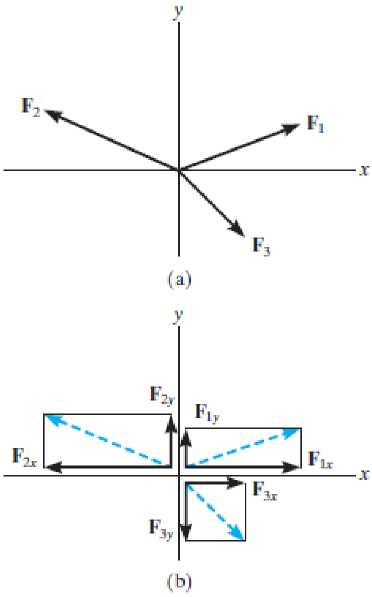


Fig. 2–17

Coplanar Force Resultants. We can use either of the two methods just described to determine the resultant of several ***coplanar forces***, i.e., forces that all lie in the same plane. To do this, each force is first resolved into its x and y components, and then the respective components are added using *scalar algebra* since they are collinear. The resultant force is then formed by adding the resultant components using the parallelogram law. For example, consider the three concurrent forces in Fig. 2–17*a*, which have x and y components shown in Fig. 2–17*b*. Using Cartesian vector notation, each force is first represented as a Cartesian vector, i.e.,

$$\mathbf{F}_1 = F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}$$
$$\mathbf{F}_2 = -F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j}$$
$$\mathbf{F}_3 = F_{3x}\,\mathbf{i} - F_{3y}\,\mathbf{j}$$

The vector resultant is therefore

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j} - F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j} + F_{3x}\,\mathbf{i} - F_{3y}\,\mathbf{j}\\ &= (F_{1x} - F_{2x} + F_{3x})\mathbf{i} + (F_{1y} + F_{2y} - F_{3y})\mathbf{j}\\ &= (F_{Rx})\mathbf{i} + (F_{Ry})\mathbf{j}\end{aligned}$$

If *scalar notation* is used, then indicating the positive directions of components along the x and y axes with symbolic arrows, we have

$$\xrightarrow{+}\qquad (F_R)_x = F_{1x} - F_{2x} + F_{3x}$$
$$+\uparrow\qquad (F_R)_y = F_{1y} + F_{2y} - F_{3y}$$

These are the *same* results as the **i** and **j** components of $\mathbf{F}_R$ determined above.

We can represent the components of the resultant force of any number of coplanar forces symbolically by the algebraic sum of the x and y components of all the forces, i.e.,

$$(F_R)_x = \Sigma F_x$$
$$(F_R)_y = \Sigma F_y \quad (2\text{–}1)$$

Once these components are determined, they may be sketched along the x and y axes with their proper sense of direction, and the resultant force can be determined from vector addition, as shown in Fig. 2–17*c*. From this sketch, the magnitude of $\mathbf{F}_R$ is then found from the Pythagorean theorem; that is,

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

Also, the angle θ, which specifies the direction of the resultant force, is determined from trigonometry:

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$

The above concepts are illustrated numerically in the examples which follow.

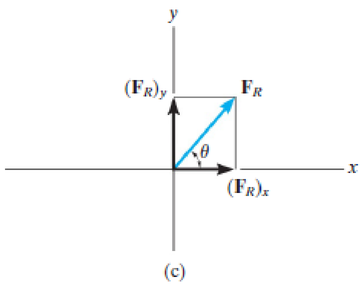


Fig. 2–17 (cont.)

Important Points

- The resultant of several coplanar forces can easily be determined if an x, y coordinate system is established and the forces are resolved along the axes.
- The direction of each force is specified by the angle its line of action makes with one of the axes, or by a slope triangle.
- The orientation of the x and y axes is arbitrary, and their positive direction can be specified by the Cartesian unit vectors **i** and **j**.
- The x and y components of the *resultant force* are simply the algebraic addition of the components of all the coplanar forces.
- The magnitude of the resultant force is determined from the Pythagorean theorem, and when the resultant components are sketched on the x and y axes, Fig. 2–17*c*, the direction θ can be determined from trigonometry.

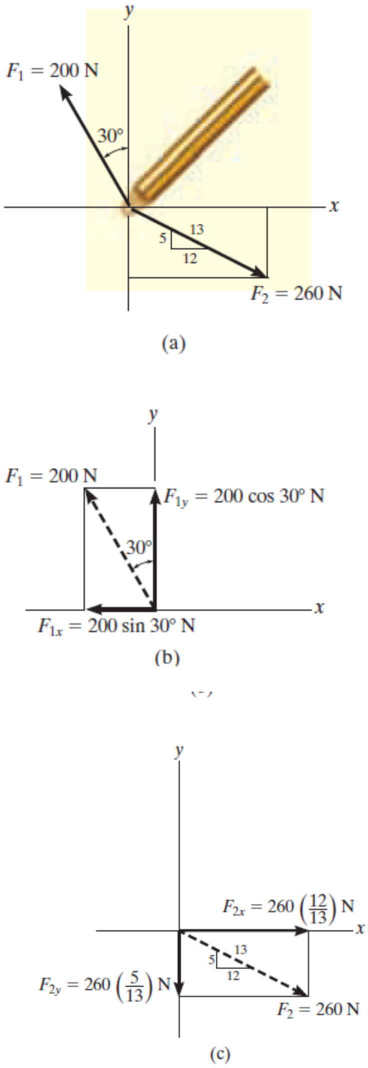


Fig. 2–18

Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom shown in Fig. 2–18*a*. Express each force as a Cartesian vector.

SOLUTION

Scalar Notation. By the parallelogram law, $\mathbf{F}_1$ is resolved into x and y components, Fig. 2–18*b*. Since $\mathbf{F}_{1x}$ acts in the $-x$ direction, and $\mathbf{F}_{1y}$ acts in the $+y$ direction, we have

$$F_{1x} = -200 \sin 30° \text{ N} = -100 \text{ N} = 100 \text{ N} \leftarrow \qquad \textit{Ans.}$$

$$F_{1y} = 200 \cos 30° \text{ N} = 173 \text{ N} = 173 \text{ N} \uparrow \qquad \textit{Ans.}$$

The force $\mathbf{F}_2$ is resolved into its x and y components, as shown in Fig. 2–18*c*. Here the *slope* of the line of action for the force is indicated. From this "slope triangle" we could obtain the angle θ, e.g., $\theta = \tan^{-1}\left(\frac{5}{12}\right)$, and then proceed to determine the magnitudes of the components in the same manner as for $\mathbf{F}_1$. The easier method, however, consists of using proportional parts of similar triangles, i.e.,

$$\frac{F_{2x}}{260 \text{ N}} = \frac{12}{13} \qquad F_{2x} = 260 \text{ N}\left(\frac{12}{13}\right) = 240 \text{ N}$$

Similarly,

$$F_{2y} = 260 \text{ N}\left(\frac{5}{13}\right) = 100 \text{ N}$$

Notice how the magnitude of the *horizontal component*, $\mathbf{F}_{2x}$, was obtained by multiplying the force magnitude by the ratio of the *horizontal leg* of the slope triangle divided by the hypotenuse; whereas the magnitude of the *vertical component*, F_{2y}, was obtained by multiplying the force magnitude by the ratio of the *vertical leg* divided by the hypotenuse. Hence, using scalar notation to represent these components, we have

$$F_{2x} = 240 \text{ N} = 240 \text{ N} \rightarrow \qquad \textit{Ans.}$$

$$F_{2y} = -100 \text{ N} = 100 \text{ N} \downarrow \qquad \textit{Ans.}$$

Cartesian Vector Notation. Having determined the magnitudes and directions of the components of each force, we can express each force as a Cartesian vector.

$$\mathbf{F}_1 = \{-100\mathbf{i} + 173\mathbf{j}\} \text{ N} \qquad \textit{Ans.}$$

$$\mathbf{F}_2 = \{240\mathbf{i} - 100\mathbf{j}\} \text{ N} \qquad \textit{Ans.}$$

EXAMPLE 2.6

The link in Fig. 2–19*a* is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.

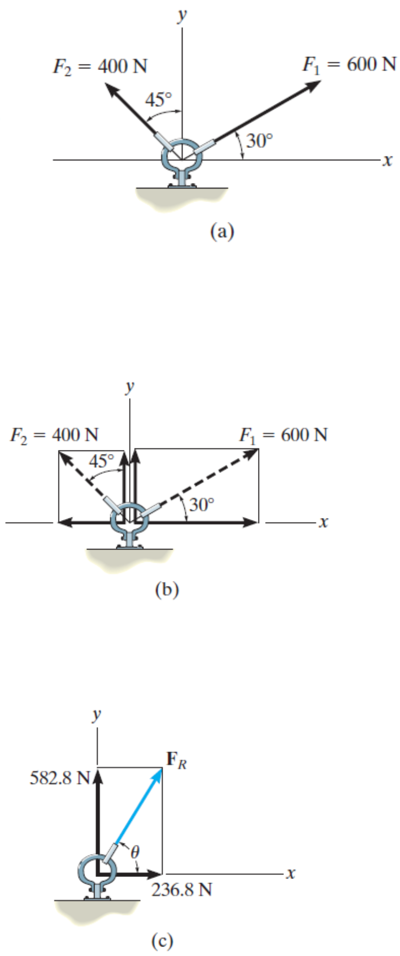


Fig. 2–19

SOLUTION I

Scalar Notation. First we resolve each force into its x and y components, Fig. 2–19*b*, then we sum these components algebraically.

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = 600 \cos 30° \text{ N} - 400 \sin 45° \text{ N} = 236.8 \text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = 600 \sin 30° \text{ N} + 400 \cos 45° \text{ N} = 582.8 \text{ N} \uparrow$$

The resultant force, shown in Fig. 2–19*c*, has a *magnitude* of

$$F_R = \sqrt{(236.8 \text{ N})^2 + (582.8 \text{ N})^2} = 629 \text{ N} \qquad \textit{Ans.}$$

From the vector addition,

$$\theta = \tan^{-1}\left(\frac{582.8 \text{ N}}{236.8 \text{ N}}\right) = 67.9° \qquad \textit{Ans.}$$

SOLUTION II

Cartesian Vector Notation. From Fig. 2–19*b*, each force is first expressed as a Cartesian vector.

$$\mathbf{F}_1 = \{600 \cos 30°\mathbf{i} + 600 \sin 30°\mathbf{j}\} \text{ N}$$
$$\mathbf{F}_2 = \{-400 \sin 45°\mathbf{i} + 400 \cos 45°\mathbf{j}\} \text{ N}$$

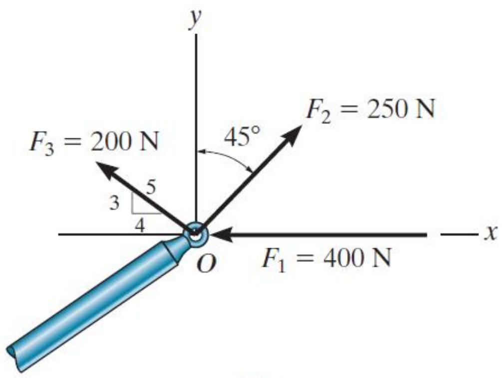
Then,

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = (600 \cos 30° \text{ N} - 400 \sin 45° \text{ N})\mathbf{i} + (600 \sin 30° \text{ N} + 400 \cos 45° \text{ N})\mathbf{j} = \{236.8\mathbf{i} + 582.8\mathbf{j}\} \text{ N}$$

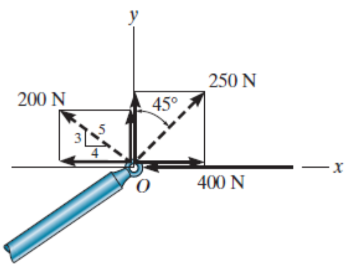
The magnitude and direction of $\mathbf{F}_R$ are determined in the same

NOTE: Comparing the two methods of solution, notice that the use of scalar notation is more efficient since the components can be found directly, without first having to express each force as a Cartesian vector before adding the components. Later, however, we will show that Cartesian vector analysis is very beneficial for solving three-dimensional problems.

The end of the boom O in Fig. 2–20a is subjected to three concurrent and coplanar forces. Determine the magnitude and direction of the resultant force.



(a)



(b)

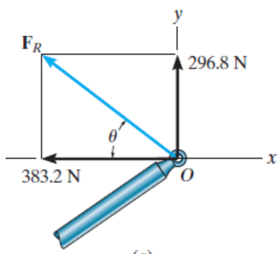
SOLUTION

Each force is resolved into its x and y components, Fig. 2–20b. Summing the x components, we have

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \quad (F_R)_x = -400 \text{ N} + 250 \sin 45° \text{ N} - 200\left(\tfrac{4}{5}\right) \text{ N}$$
$$= -383.2 \text{ N} = 383.2 \text{ N} \leftarrow$$

The negative sign indicates that F_{Rx} acts to the left, i.e., in the negative x direction, as noted by the small arrow. Obviously, this occurs because F_1 and F_3 in Fig. 2–20b contribute a greater pull to the left than F_2 which pulls to the right. Summing the y components yields

$$+\uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y = 250 \cos 45° \text{ N} + 200\left(\tfrac{3}{5}\right) \text{ N}$$
$$= 296.8 \text{ N} \uparrow$$



(c)

Fig. 2–20

The resultant force, shown in Fig. 2–20c, has a *magnitude* of

$$F_R = \sqrt{(-383.2 \text{ N})^2 + (296.8 \text{ N})^2}$$
$$= 485 \text{ N} \qquad \textit{Ans.}$$

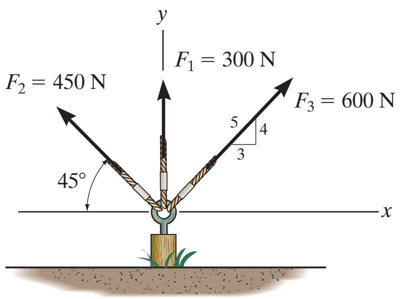
From the vector addition in Fig. 2–20c, the direction angle θ is

$$\theta = \tan^{-1}\left(\frac{296.8}{383.2}\right) = 37.8° \qquad \textit{Ans.}$$

NOTE: Application of this method is more convenient, compared to using two applications of the parallelogram law, first to add $\mathbf{F}_1$ and $\mathbf{F}_2$ then adding $\mathbf{F}_3$ to this resultant.

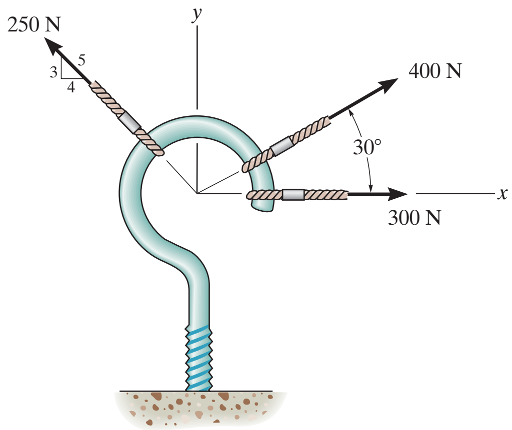
FUNDAMENTAL PROBLEMS

F2–7. Resolve each force acting on the post into its x and y components.



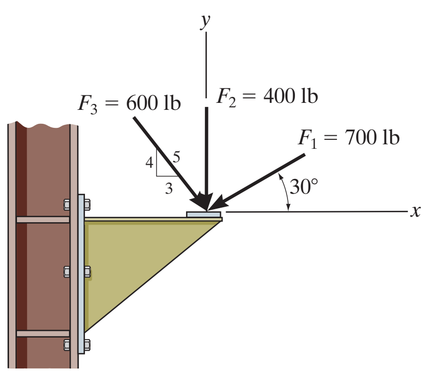
Prob. F2–7

F2–8. Determine the magnitude and direction of the resultant force.



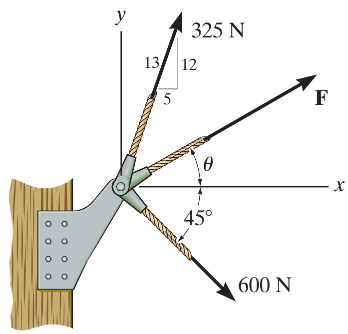
Prob. F2–8

F2–9. Determine the magnitude of the resultant force acting on the corbel and its direction θ measured counterclockwise from the x axis.



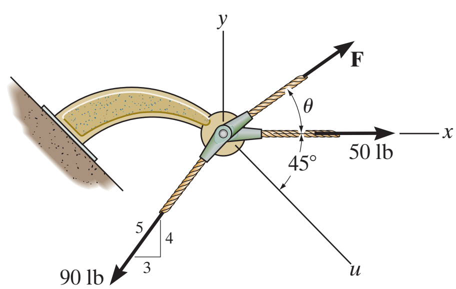
Prob. F2–9

F2–10. If the resultant force acting on the bracket is to be 750 N directed along the positive x axis, determine the magnitude of **F** and its direction θ.



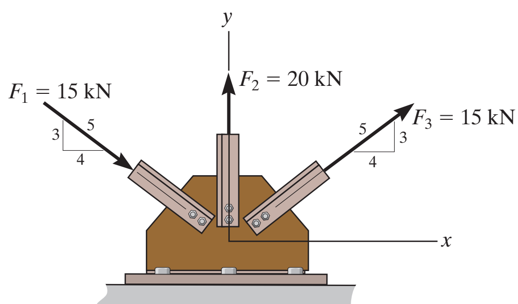
Prob. F2–10

F2–11. If the magnitude of the resultant force acting on the bracket is to be 80 lb directed along the u axis, determine the magnitude of **F** and its direction θ.



Prob. F2–11

F2–12. Determine the magnitude of the resultant force and its direction θ measured counterclockwise from the positive x axis.



Prob. F2–12

Lecture 3

2.5 Cartesian Vectors

The operations of vector algebra, when applied to solving problems in *three dimensions*, are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

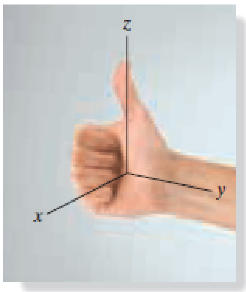


Fig. 2–21 (© Russell C. Hibbeler)

Right-Handed Coordinate System. We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be ***right-handed*** if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, Fig. 2–21.

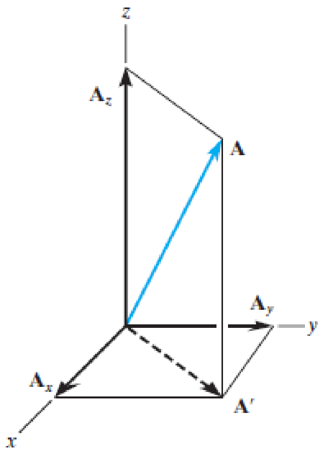


Fig. 2–22

Rectangular Components of a Vector. A vector $\mathbf{A}$ may have one, two, or three rectangular components along the x, y, z coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when $\mathbf{A}$ is directed within an octant of the x, y, z frame, Fig. 2–22, then by two successive applications of the parallelogram law, we may resolve the vector into components as $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$ and then $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$. Combining these equations, to eliminate $\mathbf{A}'$, $\mathbf{A}$ is represented by the vector sum of its *three* rectangular components,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \qquad (2\text{–}2)$$

Cartesian Unit Vectors. In three dimensions, the set of Cartesian unit vectors, **i**, **j**, **k**, is used to designate the directions of the x, y, z axes, respectively. As stated in Sec. 2–4, the *sense* (or arrowhead) of these vectors will be represented analytically by a plus or minus sign, depending on whether they are directed along the positive or negative x, y, or z axes. The positive Cartesian unit vectors are shown in Fig. 2–23.

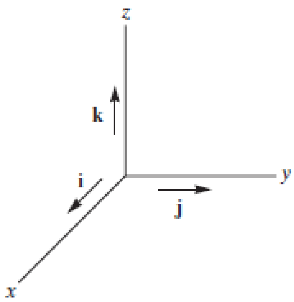


Fig. 2–23

Cartesian Vector Representation. Since the three components of **A** in Eq. 2–2 act in the positive **i**, **j**, and **k** directions, Fig. 2–24, we can write **A** in Cartesian vector form as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{2–3}$$

There is a distinct advantage to writing vectors in this manner. Separating the *magnitude* and *direction* of each *component vector* will simplify the operations of vector algebra, particularly in three dimensions.

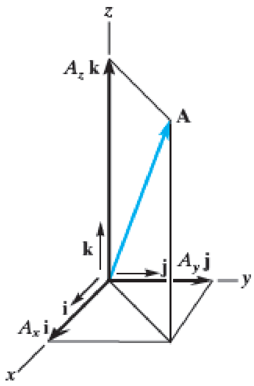


Fig. 2–24

Magnitude of a Cartesian Vector. It is always possible to obtain the magnitude of **A** provided it is expressed in Cartesian vector form. As shown in Fig. 2–25, from the blue right triangle, $A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle, $A' = \sqrt{A_x^2 + A_y^2}$. Combining these equations to eliminate A' yields

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{2–4}$$

Hence, the magnitude of **A** *is equal to the positive square root of the sum of the squares of its components.*

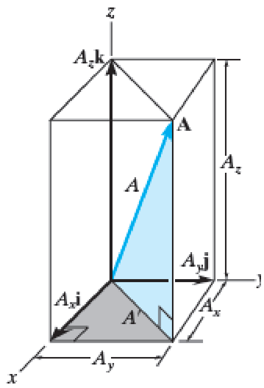


Fig. 2–25

Coordinate Direction Angles. We will define the *direction* of **A** by the ***coordinate direction angles*** α (alpha), β (beta), and γ (gamma), measured between the *tail* of **A** and the *positive* x, y, z axes provided they are located at the tail of **A**, Fig. 2–26. Note that regardless of where **A** is directed, each of these angles will be between 0° and 180°.

To determine α, β, and γ, consider the projection of **A** onto the x, y, z axes, Fig. 2–27. Referring to the colored right triangles shown in the figure, we have

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A} \tag{2–5}$$

These numbers are known as the ***direction cosines*** of **A**. Once they have been obtained, the coordinate direction angles α, β, γ can then be determined from the inverse cosines.

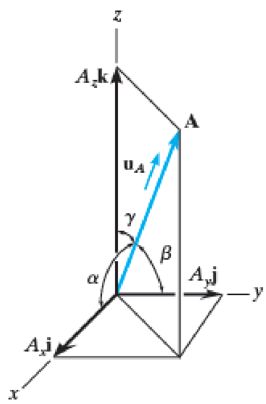


Fig. 2–26

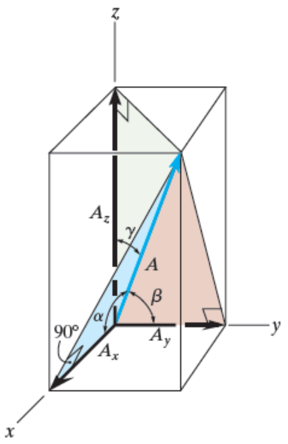


Fig. 2–27

An easy way of obtaining these direction cosines is to form a unit vector $\mathbf{u}_A$ in the direction of A, Fig. 2–26. If $\mathbf{A}$ is expressed in Cartesian vector form, $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, then $\mathbf{u}_A$ will have a magnitude of one and be dimensionless provided $\mathbf{A}$ is divided by its magnitude, i.e.,

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \qquad (2\text{–}6)$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. By comparison with Eqs. 2–5, it is seen that *the* **i**, **j**, **k** *components* of $\mathbf{u}_A$ *represent the direction cosines of* **A**, i.e.,

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \qquad (2\text{–}7)$$

Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\mathbf{u}_A$ has a magnitude of one, then from the above equation an important relation among the direction cosines can be formulated as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (2\text{–}8)$$

Here we can see that if only *two* of the coordinate angles are known, the third angle can be found using this equation.

Finally, if the magnitude and coordinate direction angles of **A** are known, then **A** may be expressed in Cartesian vector form as

$$\begin{aligned}\mathbf{A} &= A\mathbf{u}_A \\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k} \\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned} \qquad (2\text{–}9)$$

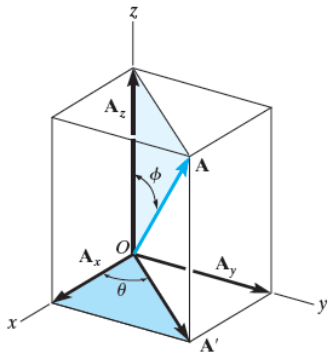


Fig. 2–28

Transverse and Azmuth Angles. Sometimes, the direction of **A** can be specified using two angles, namely, a ***transverse angle*** θ and an ***azmuth angle*** ϕ (phi), such as shown in Fig. 2–28. The components of **A** can then be determined by applying trigonometry first to the light blue right triangle, which yields

$$A_z = A\cos\phi$$

and

$$A' = A\sin\phi$$

Now applying trigonometry to the dark blue right triangle,

$$A_x = A'\cos\theta = A\sin\phi\cos\theta$$

$$A_y = A'\sin\theta = A\sin\phi\sin\theta$$

Therefore **A** written in Cartesian vector form becomes

$$\mathbf{A} = A\sin\phi\cos\theta\,\mathbf{i} + A\sin\phi\sin\theta\,\mathbf{j} + A\cos\phi\,\mathbf{k}$$

You should not memorize this equation, rather it is important to understand how the components were determined using trigonometry.

2.6 Addition of Cartesian Vectors

The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, Fig. 2–29, then the resultant vector, **R**, has components which are the scalar sums of the **i**, **j**, **k** components of **A** and **B**, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

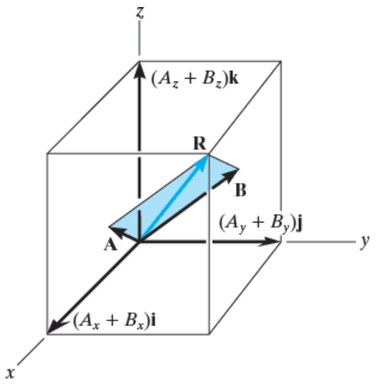
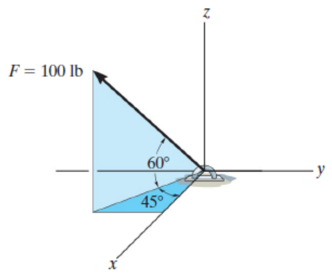


Fig. 2–29

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k} \qquad (2\text{–}10)$$

Here ΣF_x, ΣF_y, and ΣF_z represent the algebraic sums of the respective x, y, z or **i**, **j**, **k** components of each force in the system.



(a)

Express the force **F** shown in Fig. 2–30*a* as a Cartesian vector.

SOLUTION

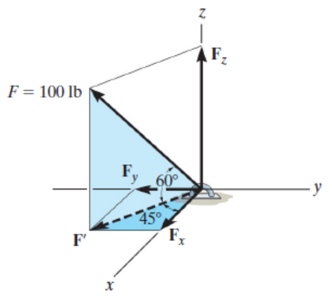
The angles of 60° and 45° defining the direction of **F** are *not* coordinate direction angles. Two successive applications of the parallelogram law are needed to resolve **F** into its x, y, z components. First $\mathbf{F} = \mathbf{F}' + \mathbf{F}_z$, then $\mathbf{F}' = \mathbf{F}_x + \mathbf{F}_y$, Fig. 2–30*b*. By trigonometry, the magnitudes of the components are

$$F_z = 100 \sin 60° \text{ lb} = 86.6 \text{ lb}$$

$$F' = 100 \cos 60° \text{ lb} = 50 \text{ lb}$$

$$F_x = F' \cos 45° = 50 \cos 45° \text{ lb} = 35.4 \text{ lb}$$

$$F_y = F' \sin 45° = 50 \sin 45° \text{ lb} = 35.4 \text{ lb}$$



(b)

Realizing that $\mathbf{F}_y$ has a direction defined by $-\mathbf{j}$, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb} \qquad \textit{Ans.}$$

To show that the magnitude of this vector is indeed 100 lb, apply Eq. 2–4,

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

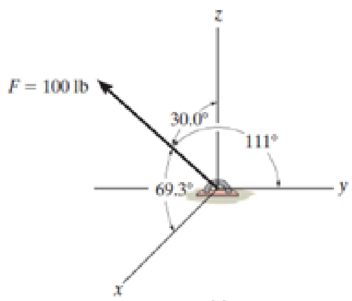
$$= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb}$$

If needed, the coordinate direction angles of **F** can be determined from the components of the unit vector acting in the direction of **F**. Hence,

$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$

$$= \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k}$$

$$= 0.354\mathbf{i} - 0.354\mathbf{j} + 0.866\mathbf{k}$$



(c)

Fig. 2–30

so that

$$\alpha = \cos^{-1}(0.354) = 69.3°$$

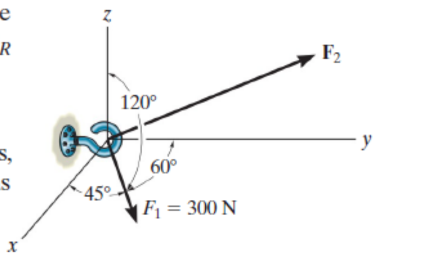
$$\beta = \cos^{-1}(-0.354) = 111°$$

$$\gamma = \cos^{-1}(0.866) = 30.0°$$

These results are shown in Fig. 2–30*c*.

EXAMPLE 2.9

Two forces act on the hook shown in Fig. 2–31*a*. Specify the magnitude of $\mathbf{F}_2$ and its coordinate direction angles so that the resultant force $\mathbf{F}_R$ acts along the positive *y* axis and has a magnitude of 800 N.



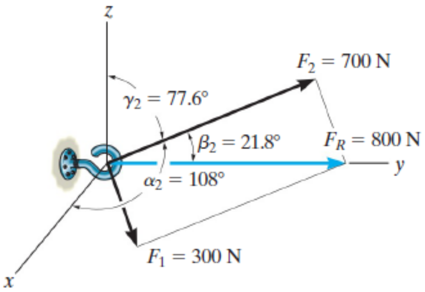


Fig. 2–31

SOLUTION

To solve this problem, the resultant force $\mathbf{F}_R$ and its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, will each be expressed in Cartesian vector form. Then, as shown in Fig. 2–31*b*, it is necessary that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$.

Applying Eq. 2–9,

$$\mathbf{F}_1 = F_1 \cos\alpha_1 \mathbf{i} + F_1 \cos\beta_1 \mathbf{j} + F_1 \cos\gamma_1 \mathbf{k}$$
$$= 300 \cos 45^\circ\, \mathbf{i} + 300 \cos 60^\circ\, \mathbf{j} + 300 \cos 120^\circ\, \mathbf{k}$$
$$= \{212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k}\}\,\text{N}$$
$$\mathbf{F}_2 = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$

Since $\mathbf{F}_R$ has a magnitude of 800 N and acts in the $+\mathbf{j}$ direction,

$$\mathbf{F}_R = (800\text{ N})(+\mathbf{j}) = \{800\mathbf{j}\}\text{ N}$$

We require

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$
$$800\mathbf{j} = 212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$
$$800\mathbf{j} = (212.1 + F_{2x})\mathbf{i} + (150 + F_{2y})\mathbf{j} + (-150 + F_{2z})\mathbf{k}$$

To satisfy this equation the **i**, **j**, **k** components of $\mathbf{F}_R$ must be equal to the corresponding **i**, **j**, **k** components of $(\mathbf{F}_1 + \mathbf{F}_2)$. Hence,

$$0 = 212.1 + F_{2x} \qquad F_{2x} = -212.1\text{ N}$$
$$800 = 150 + F_{2y} \qquad F_{2y} = 650\text{ N}$$
$$0 = -150 + F_{2z} \qquad F_{2z} = 150\text{ N}$$

The magnitude of $\mathbf{F}_2$ is thus

$$F_2 = \sqrt{(-212.1\text{ N})^2 + (650\text{ N})^2 + (150\text{ N})^2}$$
$$= 700\text{ N} \qquad \textit{Ans.}$$

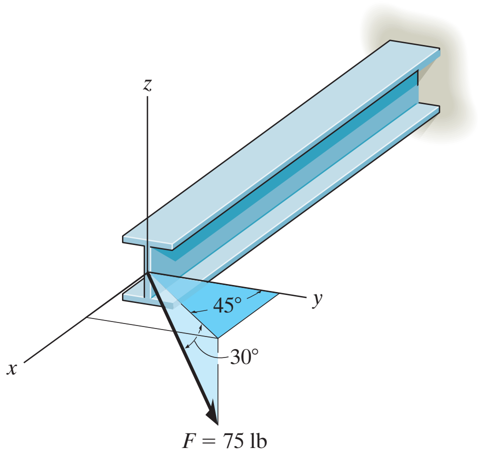
We can use Eq. 2–9 to determine $\alpha_2, \beta_2, \gamma_2$.

$$\cos\alpha_2 = \frac{-212.1}{700}; \qquad \alpha_2 = 108^\circ \qquad \textit{Ans.}$$
$$\cos\beta_2 = \frac{650}{700}; \qquad \beta_2 = 21.8^\circ \qquad \textit{Ans.}$$
$$\cos\gamma_2 = \frac{150}{700}; \qquad \gamma_2 = 77.6^\circ \qquad \textit{Ans.}$$

These results are shown in Fig. 2–31*b*.

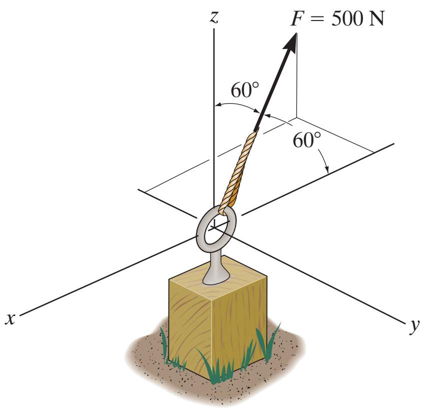
FUNDAMENTAL PROBLEMS

F2–13. Determine the coordinate direction angles of the force.



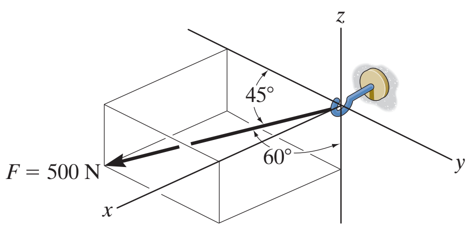
Prob. F2–13

F2–14. Express the force as a Cartesian vector.



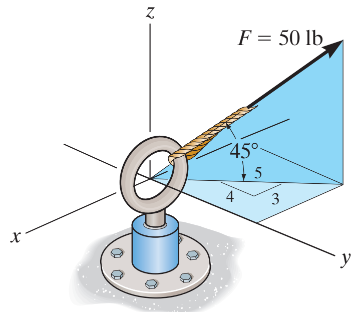
Prob. F2–14

F2–15. Express the force as a Cartesian vector.



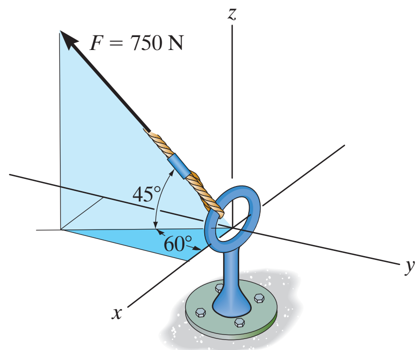
Prob. F2–15

F2–16. Express the force as a Cartesian vector.



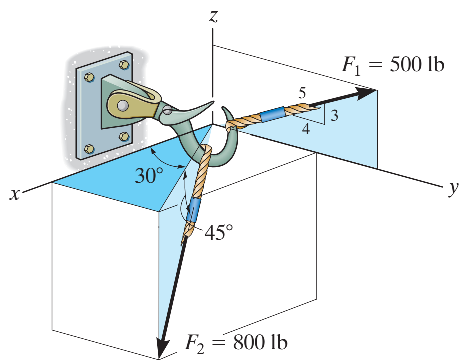
Prob. F2–16

F2–17. Express the force as a Cartesian vector.



Prob. F2–17

F2–18. Determine the resultant force acting on the hook.



Prob. F2–18

2.7 Position Vectors

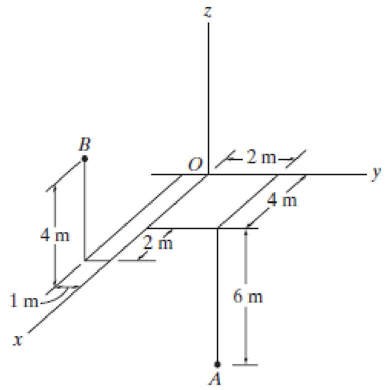


Fig. 2–32

In this section we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

x, y, z Coordinates. Throughout the book we will use a *right-handed* coordinate system to reference the location of points in space. We will also use the convention followed in many technical books, which requires the positive *z* axis to be directed *upward* (the zenith direction) so that it measures the height of an object or the altitude of a point. The *x, y* axes then lie in the horizontal plane, Fig. 2–32. Points in space are located relative to the origin of coordinates, *O*, by successive measurements along the *x, y, z* axes. For example, the coordinates of point *A* are obtained by starting at *O* and measuring $x_A = +4$ m along the *x* axis, then $y_A = +2$ m along the *y* axis, and finally $z_A = -6$ m along the *z* axis, so that *A*(4 m, 2 m, – 6 m). In a similar manner, measurements along the *x, y, z* axes from *O* to *B* yield the coordinates of *B*, that is, *B*(6 m, –1 m, 4 m).

Position Vector. A ***position vector*** **r** is defined as a fixed vector which locates a point in space relative to another point. For example, if **r** extends from the origin of coordinates, *O*, to point *P*(*x, y, z*), Fig. 2–33*a*, then **r** can be expressed in Cartesian vector form as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Note how the head-to-tail vector addition of the three components yields vector **r**, Fig. 2–33*b*. Starting at the origin *O*, one "travels" *x* in the +**i** direction, then *y* in the +**j** direction, and finally *z* in the +**k** direction to arrive at point *P*(*x, y, z*).

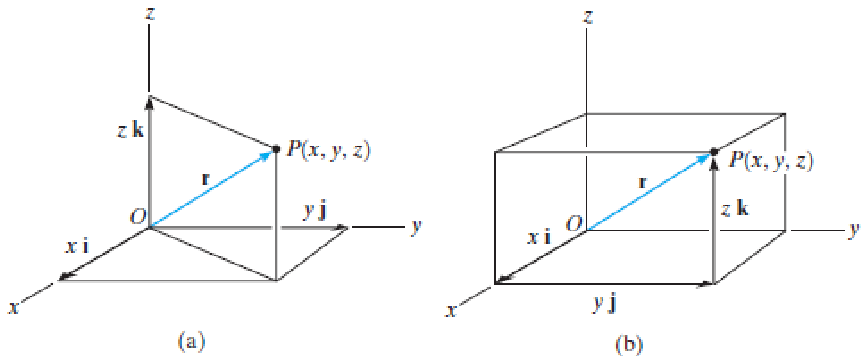
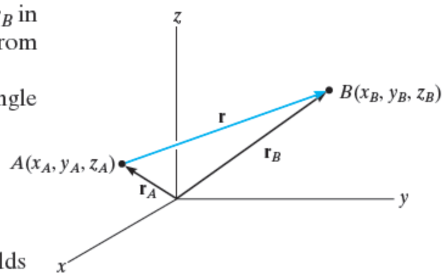


Fig. 2–33

In the more general case, the position vector may be directed from point A to point B in space, Fig. 2–34*a*. This vector is also designated by the symbol **r**. As a matter of convention, we will *sometimes* refer to this vector with *two subscripts* to indicate from and to the point where it is directed. Thus, **r** can also be designated as $\mathbf{r}_{AB}$. Also, note that $\mathbf{r}_A$ and $\mathbf{r}_B$ in Fig. 2–34*a* are referenced with only one subscript since they extend from the origin of coordinates.

From Fig. 2–34*a*, by the head-to-tail vector addition, using the triangle rule, we require



(a)

$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$

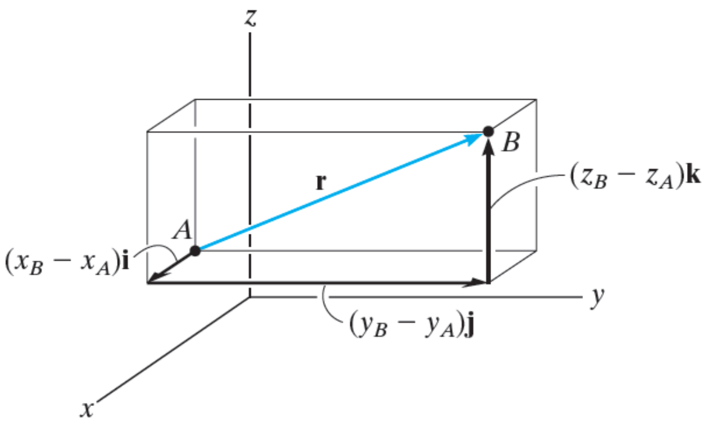
Solving for **r** and expressing $\mathbf{r}_A$ and $\mathbf{r}_B$ in Cartesian vector form yields

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

or

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} \qquad (2\text{–}11)$$

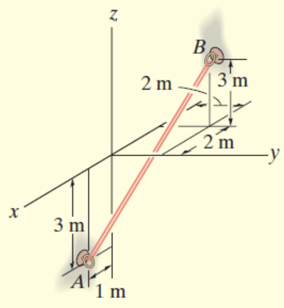
Thus, the **i**, **j**, **k** *components of the position vector* **r** *may be formed by taking the coordinates of the tail of the vector* $A(x_A, y_A, z_A)$ *and subtracting them from the corresponding coordinates of the head* $B(x_B, y_B, z_B)$. We can also form these components *directly*, Fig. 2–34*b*, by starting at A and moving through a distance of $(x_B - x_A)$ along the positive x axis $(+\mathbf{i})$, then $(y_B - y_A)$ along the positive y axis $(+\mathbf{j})$, and finally $(z_B - z_A)$ along the positive z axis $(+\mathbf{k})$ to get to B.



(b)

Fig. 2–34

EXAMPLE 2.10



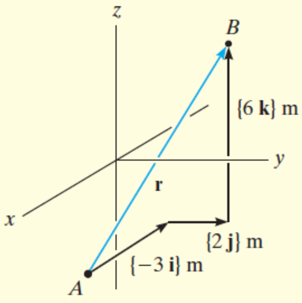
(a)

An elastic rubber band is attached to points A and B as shown in Fig. 2–35*a*. Determine its length and its direction measured from A toward B.

SOLUTION

We first establish a position vector from A to B, Fig. 2–35*b*. In accordance with Eq. 2–11, the coordinates of the tail $A(1\text{ m}, 0, -3\text{ m})$ are subtracted from the coordinates of the head $B(-2\text{ m}, 2\text{ m}, 3\text{ m})$, which yields

$$\mathbf{r} = [-2\text{ m} - 1\text{ m}]\mathbf{i} + [2\text{ m} - 0]\mathbf{j} + [3\text{ m} - (-3\text{ m})]\mathbf{k}$$
$$= \{-3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\}\text{ m}$$



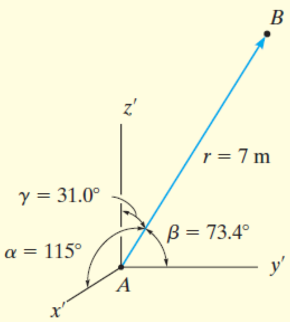
(b)

These components of **r** can also be determined *directly* by realizing that they represent the direction and distance one must travel along each axis in order to move from A to B, i.e., along the x axis $\{-3\mathbf{i}\}$ m, along the y axis $\{2\mathbf{j}\}$ m, and finally along the z axis $\{6\mathbf{k}\}$ m.

The length of the rubber band is therefore

$$r = \sqrt{(-3\text{ m})^2 + (2\text{ m})^2 + (6\text{ m})^2} = 7\text{ m} \qquad \textit{Ans.}$$

Formulating a unit vector in the direction of **r**, we have



(c)

Fig. 2–35

$$\mathbf{u} = \frac{\mathbf{r}}{r} = -\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

The components of this unit vector give the coordinate direction angles

$$\alpha = \cos^{-1}\left(-\frac{3}{7}\right) = 115^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{2}{7}\right) = 73.4^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31.0^\circ \qquad \textit{Ans.}$$

NOTE: These angles are measured from the *positive axes* of a localized coordinate system placed at the tail of **r**, as shown in Fig. 2–35*c*.

2.8 Force Vector Directed Along a Line

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Fig. 2–36, where the force **F** is directed along the cord AB. We can formulate **F** as a Cartesian vector by realizing that it has the *same direction* and *sense* as the position vector **r** directed from point A to point B on the cord. This common direction is specified by the ***unit vector*** $\mathbf{u} = \mathbf{r}/r$. Hence,

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Although we have represented **F** symbolically in Fig. 2–36, note that it has *units of force*, unlike **r**, which has units of length.

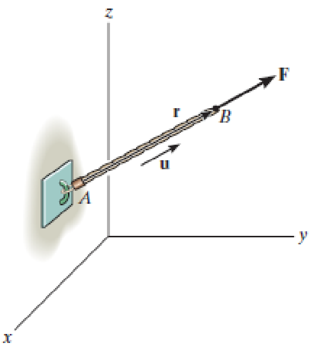
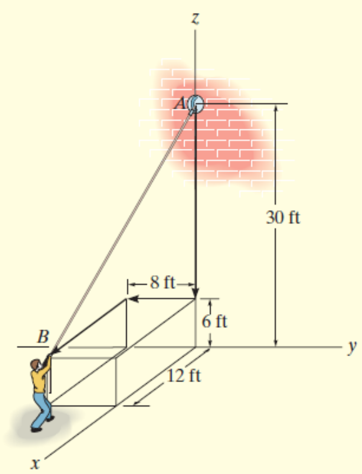


Fig. 2–36

EXAMPLE 2.11



(a)

The man shown in Fig. 2–37*a* pulls on the cord with a force of 70 lb. Represent this force acting on the support A as a Cartesian vector and determine its direction.

SOLUTION

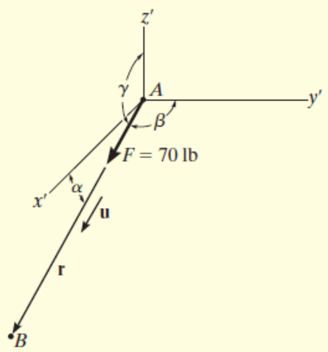
Force **F** is shown in Fig. 2–37*b*. The *direction* of this vector, **u**, is determined from the position vector **r**, which extends from A to B. Rather than using the coordinates of the end points of the cord, **r** can be determined *directly* by noting in Fig. 2–37*a* that one must travel from A $\{-24\mathbf{k}\}$ ft, then $\{-8\mathbf{j}\}$ ft, and finally $\{12\mathbf{i}\}$ ft to get to B. Thus,

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\}\ \text{ft}$$

The magnitude of **r**, which represents the *length* of cord AB, is

$$r = \sqrt{(12\ \text{ft})^2 + (-8\ \text{ft})^2 + (-24\ \text{ft})^2} = 28\ \text{ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have



(b)

Fig. 2–37

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F** has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70\ \text{lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\}\ \text{lb} \qquad \textit{Ans.}$$

The coordinate direction angles are measured between **r** (or **F**) and the *positive axes* of a localized coordinate system with origin placed at A, Fig. 2–37*b*. From the components of the unit vector:

$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6^\circ \qquad \textit{Ans.}$$

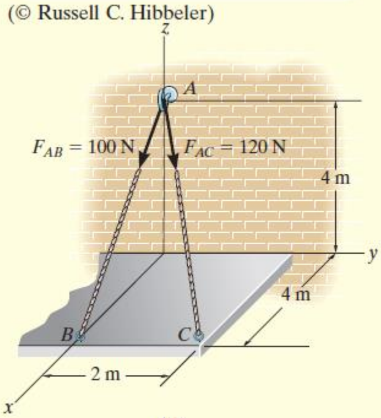
$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149^\circ \qquad \textit{Ans.}$$

NOTE: These results make sense when compared with the angles identified in Fig. 2–37*b*.

EXAMPLE 2.12

(© Russell C. Hibbeler)



(a)

The roof is supported by cables as shown in the photo. If the cables exert forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the wall hook at A as shown in Fig. 2–38*a*, determine the resultant force acting at A. Express the result as a Cartesian vector.

SOLUTION

The resultant force $\mathbf{F}_R$ is shown graphically in Fig. 2–38*b*. We can express this force as a Cartesian vector by first formulating $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ as Cartesian vectors and then adding their components. The directions of $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ are specified by forming unit vectors $\mathbf{u}_{AB}$ and $\mathbf{u}_{AC}$ along the cables. These unit vectors are obtained from the associated position vectors $\mathbf{r}_{AB}$ and $\mathbf{r}_{AC}$. With reference to Fig. 2–38*a*, to go from A to B, we must travel $\{-4\mathbf{k}\}$ m, and then $\{4\mathbf{i}\}$ m. Thus,

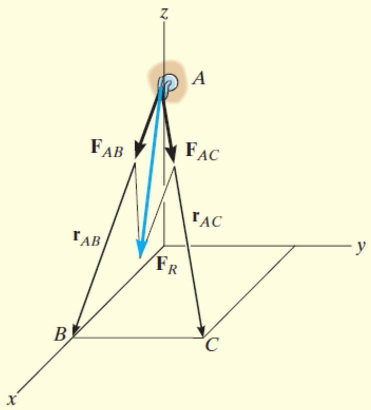
$$\mathbf{r}_{AB} = \{4\mathbf{i} - 4\mathbf{k}\} \text{ m}$$

$$r_{AB} = \sqrt{(4 \text{ m})^2 + (-4 \text{ m})^2} = 5.66 \text{ m}$$

$$\mathbf{F}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = (100 \text{ N})\left(\frac{4}{5.66}\mathbf{i} - \frac{4}{5.66}\mathbf{k}\right)$$

$$\mathbf{F}_{AB} = \{70.7\mathbf{i} - 70.7\mathbf{k}\} \text{ N}$$

To go from A to C, we must travel $\{-4\mathbf{k}\}$ m, then $\{2\mathbf{j}\}$ m, and finally $\{4\mathbf{i}\}$. Thus,



(b)

Fig. 2–38

$$\mathbf{r}_{AC} = \{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\} \text{ m}$$

$$r_{AC} = \sqrt{(4 \text{ m})^2 + (2 \text{ m})^2 + (-4 \text{ m})^2} = 6 \text{ m}$$

$$\mathbf{F}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = (120 \text{ N})\left(\frac{4}{6}\mathbf{i} + \frac{2}{6}\mathbf{j} - \frac{4}{6}\mathbf{k}\right)$$

$$= \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\} \text{ N}$$

The resultant force is therefore

$$\mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC} = \{70.7\mathbf{i} - 70.7\mathbf{k}\} \text{ N} + \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\} \text{ N}$$

$$= \{151\mathbf{i} + 40\mathbf{j} - 151\mathbf{k}\} \text{ N} \qquad \textit{Ans.}$$

EXAMPLE 2.13

The force in Fig. 2–39*a* acts on the hook. Express it as a Cartesian vector.

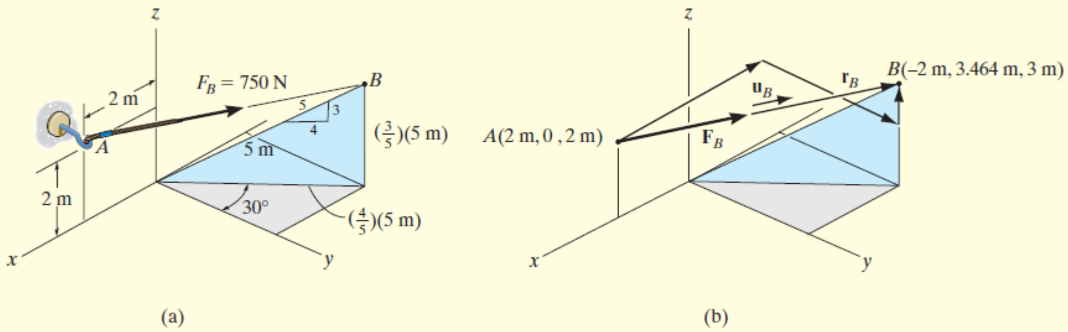


Fig. 2–39

SOLUTION

As shown in Fig. 2–39*b*, the coordinates for points *A* and *B* are

$$A(2\text{ m}, 0, 2\text{ m})$$

and

$$B\left[-\left(\frac{4}{5}\right)5\sin 30^\circ\text{ m},\ \left(\frac{4}{5}\right)5\cos 30^\circ\text{ m},\ \left(\frac{3}{5}\right)5\text{ m}\right]$$

or

$$B(-2\text{ m}, 3.464\text{ m}, 3\text{ m})$$

Therefore, to go from *A* to *B*, one must travel $\{-4\mathbf{i}\}$ m, then $\{3.464\mathbf{j}\}$ m, and finally $\{1\mathbf{k}\}$ m. Thus,

$$\mathbf{u}_B = \left(\frac{\mathbf{r}_B}{r_B}\right) = \frac{\{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}\}\text{ m}}{\sqrt{(-4\text{ m})^2 + (3.464\text{ m})^2 + (1\text{ m})^2}}$$

$$= -0.7428\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k}$$

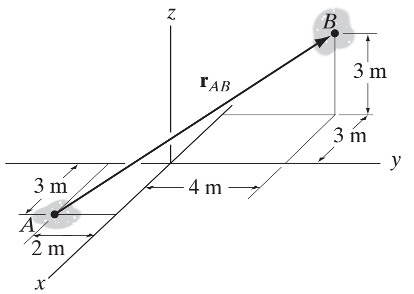
Force $\mathbf{F}_B$ expressed as a Cartesian vector becomes

$$\mathbf{F}_B = F_B\,\mathbf{u}_B = (750\text{ N})(-0.74281\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k})$$

$$= \{-557\mathbf{i} + 482\mathbf{j} + 139\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

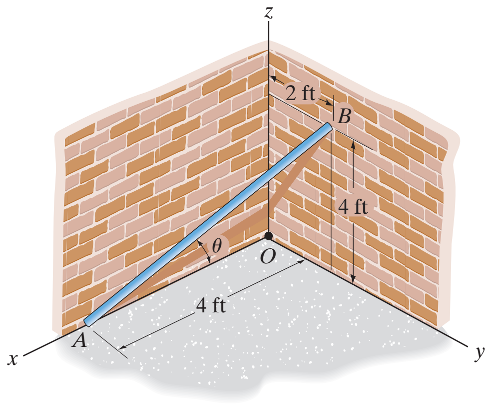
FUNDAMENTAL PROBLEMS

F2–19. Express the position vector $\mathbf{r}_{AB}$ in Cartesian vector form, then determine its magnitude and coordinate direction angles.



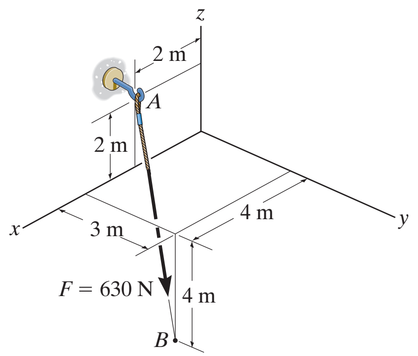
Prob. F2–19

F2–20. Determine the length of the rod and the position vector directed from A to B. What is the angle θ?



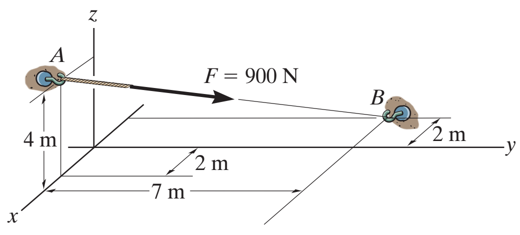
Prob. F2–20

F2–21. Express the force as a Cartesian vector.



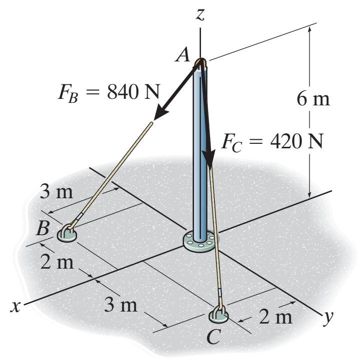
Prob. F2–21

F2–22. Express the force as a Cartesian vector.



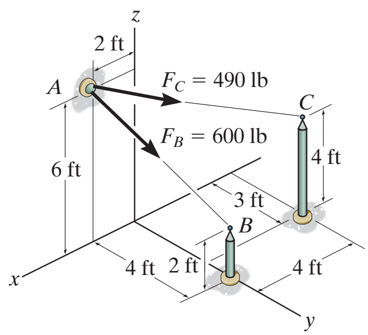
Prob. F2–22

F2–23. Determine the magnitude of the resultant force at A.



Prob. F2–23

F2–24. Determine the resultant force at A.



Prob. F2–24

Lecture 4

Force System

4.1 Moment of Force

When a force is applied to a body it will produce a tendency for the body to rotate about a point is not on the line action of the force. This tendency to rotate is sometimes called a **torque,** but most often it is called **the moment of a force** or simply **moment.**

The **magnitude** of the moment is:

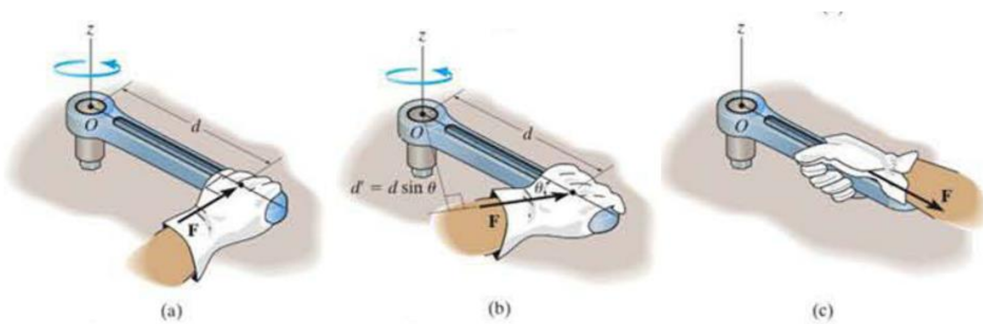
$$M = F\,d \quad .. (4.1)$$

Where d is the ***moment arm*** *or perpendicular distance* from the axis at point O to the line of action of the force.

Units of moment magnitude consist of force times distance, i.e., N.m or lb.ft.

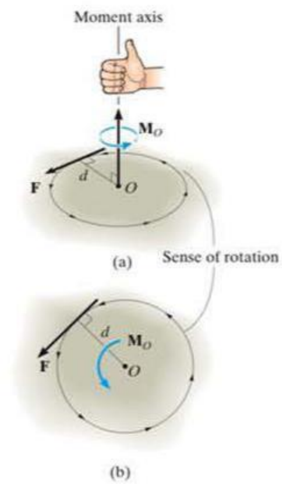
***Note* 1:** If the force $\boldsymbol{F}$ is applied at an angle $\theta \neq 90^o$, Figure b, then it will be difficult to turn the bolt since the moment arm $\boldsymbol{d' = d \sin\theta}$ will be smaller than d.

***Note* 2:** If $\boldsymbol{F}$ is applied along the wrench, Figure c, its moment arm will be zero since the line of action of $\boldsymbol{F}$ will intersect point O (the z-axis). As a result, the moment of $\boldsymbol{F}$ about O is also zero and no turning can occur

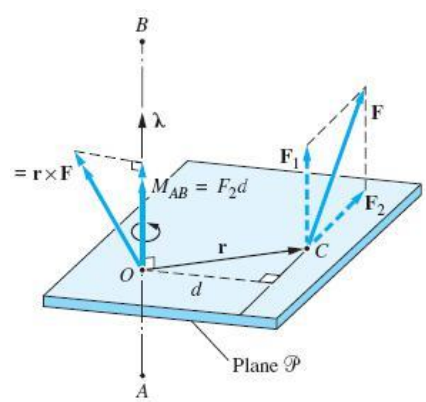


The moment $\boldsymbol{M_O}$ about O, or about an axis passing through O and perpendicular to the plane, is a ***vector quantity*** since it has a specified magnitude and direction.

Direction: The direction of $\boldsymbol{M_o}$ is defined by its *moment axis*, which is perpendicular to the plane that contain the force $\boldsymbol{F}$ and its moment arm $\boldsymbol{d}$. The right-hand rule is used to establish the sense of direction of $\boldsymbol{M_o}$. According to this rule, the nature curl of the fingers of the right-hand, as they are drawn towards the palm, represent the tendency for rotation caused by the moment. As the action is performed, the thumb of the right-hand will give the direction sense of $\boldsymbol{M_o}$. Notice that the moment vector is represented in three-directionally by a curl around an arrow as in Figure b. Since in this case the moment will tend to cause a counterclockwise rotation, the moment vector is actually directed out of page.



If the force does not lie in a plane perpendicular to the moment axis, it may be resolved into two components, one being parallel to the moment axis and the other lying in a plane perpendicular to the axis. The component of parallel to the reference axis has no tendency to rotate the body about the axis and has no moment with respect to this axis. The moment of the other component is thus the moment of the force with respect to the line or axis.

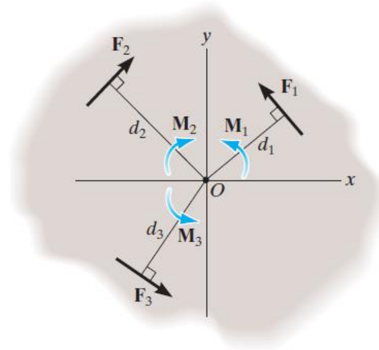


$$M_{AB} = F_2 d \qquad .. (4.2)$$

Resultant Moment: For two-dimensional problems, where all the forces lie in the *x-y* plane the resultant $(\boldsymbol{M_R})_o$ about point *O* (the *z*-axis) can be determined by *finding the algebraic sum* of the moments caused by all forces in the system. As a convention, we will generally consider *positive moment* as *counterclockwise* since they are directed along the positive *z*-axis (out of page). *Clockwise moment* will be *negative*. Doing this, the directional sense of each moment can be represented by a *plus* or *minus* sign.

Using this sign convention, the resultant moment in figure below is therefore:

$$\circlearrowleft +(M_R)_o = \Sigma Fd; \qquad (M_R)_o = F_1d_1 - F_2d_2 + F_3d_3 \text{ .. (4.3)}$$



If the numerical result of this sum is positive scalar, $(M_R)_o$ will be counterclockwise moment (out of page); if the result is negative, $(M_R)_o$ will be clockwise moment (into the page).

Example 4.1: For each case illustrated in Figures below, determine the moment of the force about point *O*.

Solution:

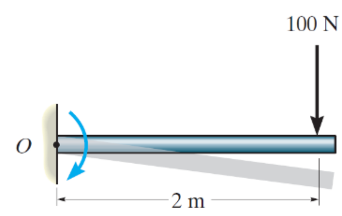
Fig. *a* $M_O = (100 \text{ N})(2 \text{ m}) = 200 \text{ N}\cdot\text{m} \circlearrowright$

Fig. *b* $M_O = (50 \text{ N})(0.75 \text{ m}) = 37.5 \text{ N}\cdot\text{m} \circlearrowright$

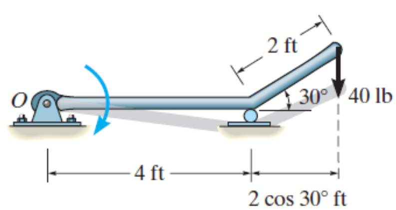
Fig. *c* $M_O = (40 \text{ lb})(4 \text{ ft} + 2\cos 30^\circ \text{ ft}) = 229 \text{ lb}\cdot\text{ft} \circlearrowright$

Fig. *d* $M_O = (60 \text{ lb})(1 \sin 45^\circ \text{ ft}) = 42.4 \text{ lb}\cdot\text{ft} \circlearrowleft$

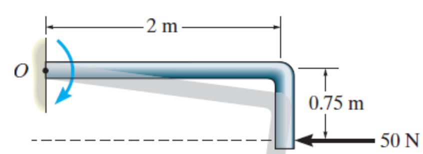
Fig. *e* $M_O = (7 \text{ kN})(4 \text{ m} - 1 \text{ m}) = 21.0 \text{ kN}\cdot\text{m} \circlearrowleft$



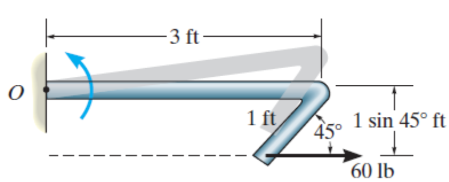
(a)



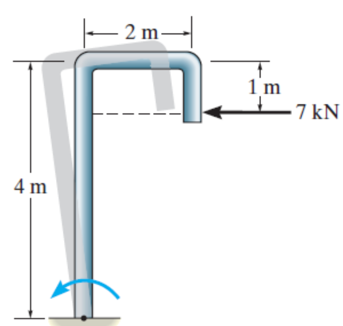
(c)



(b)

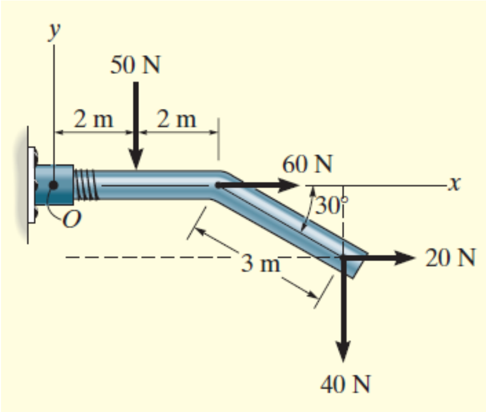


(d)



(e)

Example 4.2: Determine the resultant moment of the four forces acting on the rod shown in Figure about *O*.



SOLUTION

Assuming that positive moments act in the $+\mathbf{k}$ direction, i.e., counterclockwise, we have

$$\circlearrowleft + (M_R)_o = \Sigma Fd;$$

$$(M_R)_o = -50\text{ N}(2\text{ m}) + 60\text{ N}(0) + 20\text{ N}(3\sin 30^\circ\text{ m})$$

$$-40\text{ N}(4\text{ m} + 3\cos 30^\circ\text{ m})$$

$$(M_R)_o = -334\text{ N}\cdot\text{m} = 334\text{ N}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.

4.2 Cross Product

The moment of a force will be formulated using Cartesian vectors in the next section. Before doing this, however, it is first necessary to expand our knowledge of vector algebra and introduce the **cross-product method** of vector multiplication, first used by Willard Gibbs in lectures given in the late 19th century.

The **cross product** of two vectors **A** and **B** yields the vector **C**, which is written :

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

Magnitude:

The magnitude of C is defined as the product of the magnitudes of A and B and the sine of the angle θ between their tails ($0^o \leq \theta \leq 180^o$). Thus,

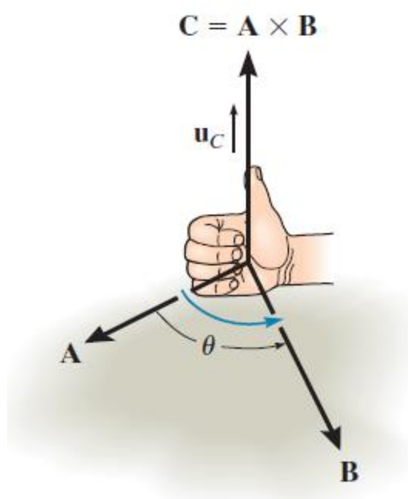
$$\mathbf{C} = AB \sin \theta$$

Direction.

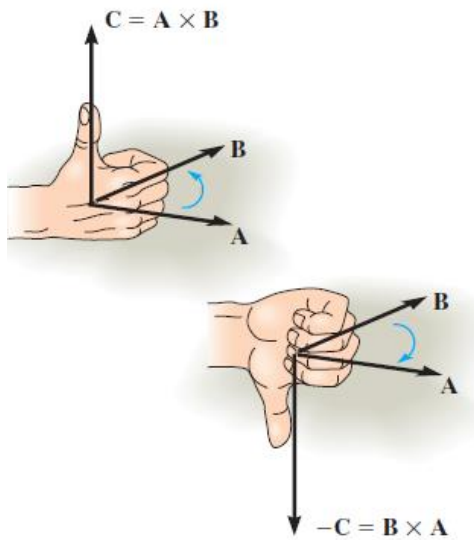
Vector C has a direction that is perpendicular to the plane containing A and B such that C is specified by **the right-hand rule**; i.e., curling the fingers of the right hand from vector A (cross) to vector B, the thumb points in the direction of C, as shown in Fig. Knowing both the magnitude and direction of **C**, we can write

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C$$

where the **scalar** AB sin θ defines the **magnitude** of C and the **unit vector** $\mathbf{u}_C$ defines the **direction** of **C**.



4.2.1 Laws of Operation



The commutative law is not valid; i.e., A x B ≠ B x A. Rather,

$$A \times B = -B \times A$$

This is shown in Fig. by using **the right-hand rule**. The cross product B x A yields a vector that has the same magnitude but acts in the opposite direction to C; i.e., B x A = - C.

- If the cross product is multiplied by a scalar **a**, it obeys the associative law;

$$a(A \times B) = (aA) \times B = A \times (aB) = (A \times B)a$$

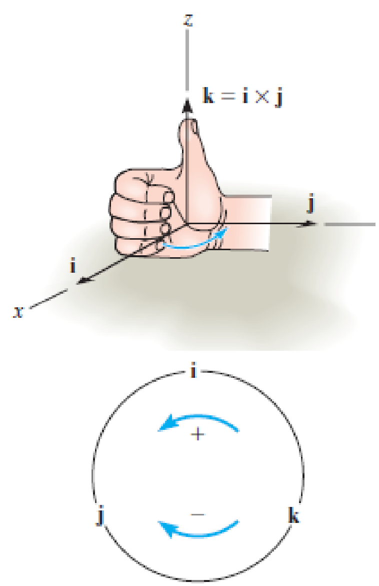
- The vector cross product also obeys the distributive law of addition,

$$A \times (B + D) = (A \times B) + (A \times D)$$

4.2.2 Cartesian Vector Formulation

To find the cross product of any pair of Cartesian unit vectors using the right-hand rule. As shown in Fig.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{i} \times \mathbf{i} = 0$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{j} = 0$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{k} \times \mathbf{k} = 0$$



Let us now consider the cross product of two general vectors A and B which are expressed in Cartesian vector form. We have

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})\\ &= A_xB_x(\mathbf{i}\times\mathbf{i})+A_xB_y(\mathbf{i}\times\mathbf{j})+A_xB_z(\mathbf{i}\times\mathbf{k})\\ &\quad + A_yB_x(\mathbf{j}\times\mathbf{i})+A_yB_y(\mathbf{j}\times\mathbf{j})+A_yB_z(\mathbf{j}\times\mathbf{k})\\ &\quad + A_zB_x(\mathbf{k}\times\mathbf{i})+A_zB_y(\mathbf{k}\times\mathbf{j})+A_zB_z(\mathbf{k}\times\mathbf{k})\end{aligned}$$

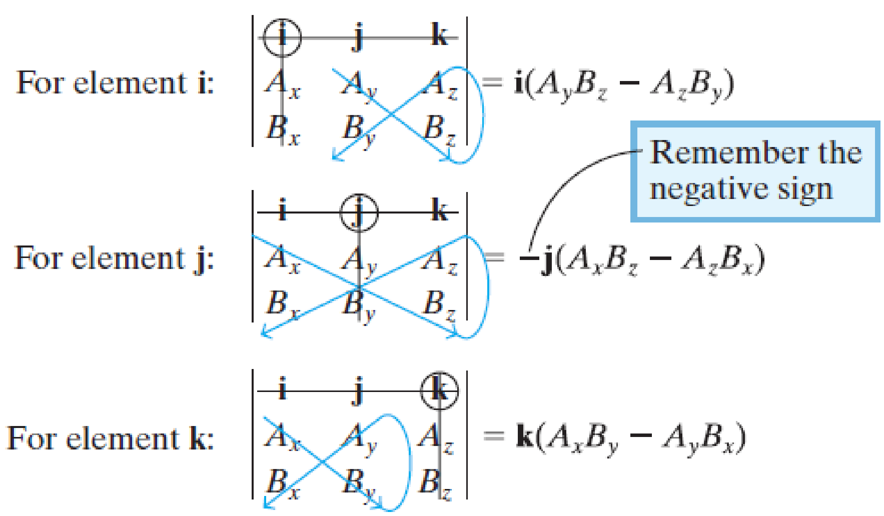
Carrying out the cross-product operations and combining terms yields

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \quad \text{... (4.4)}$$

This equation may also be written in a more compact determinant form as

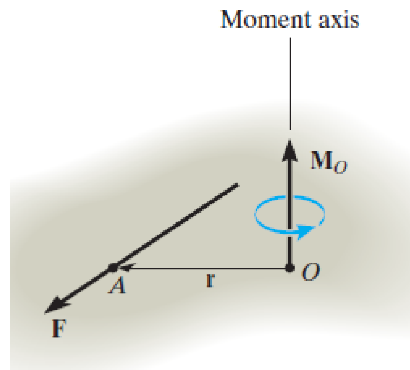
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

The three minors in equation 4.4 can be generated in accordance with the following scheme:



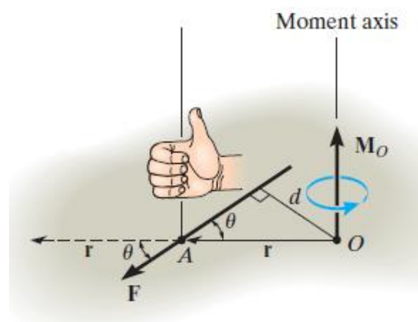
4.3 Moment of a Force—Vector Formulation

The moment of a force F about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and F, *Fig.a*, can be expressed using the vector cross product $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.



(a)

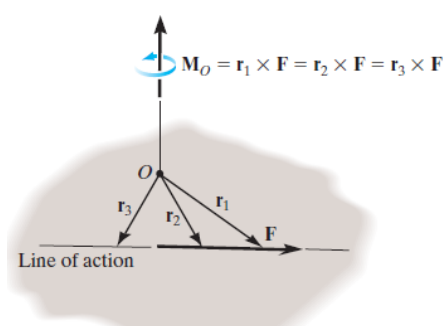
Magnitude. The magnitude of the cross product is defined as $M_O = rF\sin\theta$, where the angle **θ** is measured between the tails of r and F. To establish this angle, r must be treated as a sliding vector so that u can be constructed properly, *Fig.b*. Since the moment arm $d = r \sin \theta$, then $M_O = Fd$.



(b)

Direction. The direction and sense of M_O determined by the right-hand rule as it applies to the cross product. Thus, sliding r to the dashed position and curling the right-hand fingers from r toward F, "r cross F," the thumb is directed upward or perpendicular to the plane containing r and F and this is in the same direction as M_O, the moment of the force about point O, *Fig.b* is $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.

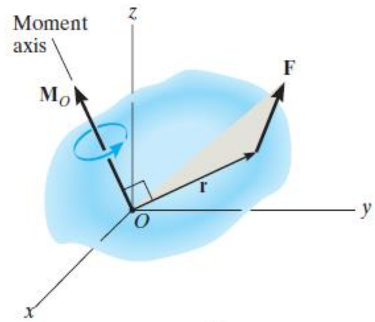
Principle of Transmissibility. The cross product operation is often used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words, we can use any position vector r measured from point O to any point on the line of action of the force F, Fig. Thus,



$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F} .$$

Cartesian Vector Formulation. If we establish x, y, z coordinate axes, then the position vector r and force F can be expressed as Cartesian vectors, Fig. a Applying Eq. 4–5 we have :

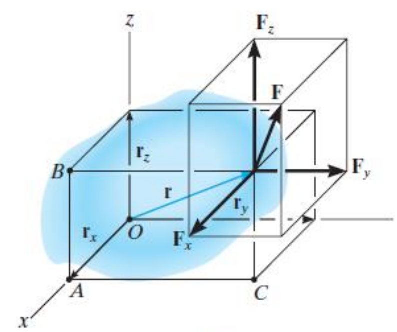
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



(a)

r_x, r_y, r_z represent the *x, y, z* components of the position vector drawn from point *O* to *any point* on the line of action of the force

F_x, F_y, F_z represent the *x, y, z* components of the force vector



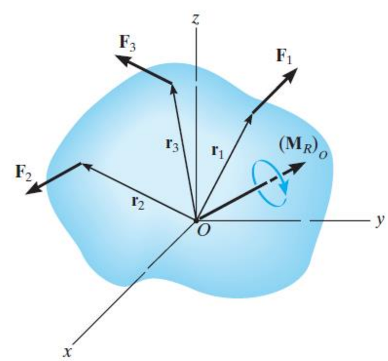
(b)

If the determinant is expanded, then like Eq. 4–4 we have

$$\mathbf{M}_O = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k}$$

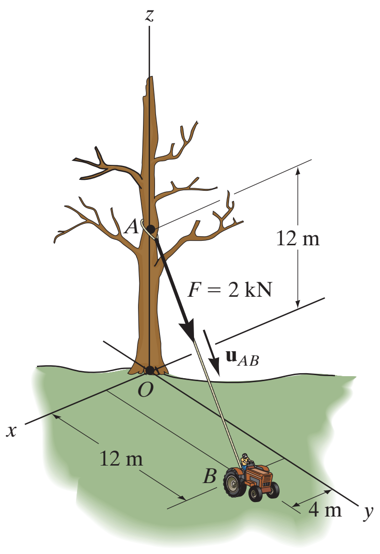
Resultant Moment of a System of Forces. If a body is acted upon by a system of forces, Fig. 4–13, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as

$$(\mathbf{M}_R)_O = \sum(\mathbf{r} \times \mathbf{F})$$



EXAMPLE 4.3

Determine the moment produced by the force **F** in Fig. 4–14*a* about point O. Express the result as a Cartesian vector.



(a)

SOLUTION

As shown in Fig. 4–14*b*, either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point O. These position vectors are

$$\mathbf{r}_A = \{12\mathbf{k}\}\text{ m} \quad\text{and}\quad \mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}\text{ m}$$

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\text{ kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\text{ m}}{\sqrt{(4\text{ m})^2 + (12\text{ m})^2 + (-12\text{ m})^2}}\right]$$

$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\text{ kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$$

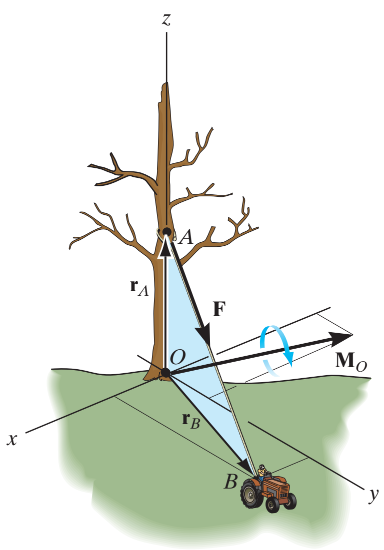
$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

or

$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



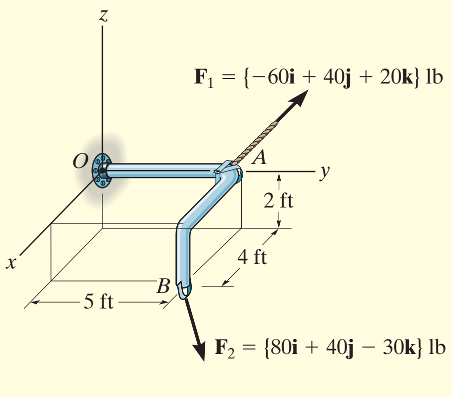
(b)

Fig. 4–14

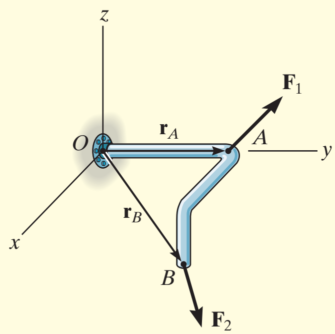
NOTE: As shown in Fig. 4–14*b*, $\mathbf{M}_O$ acts perpendicular to the plane that contains $\mathbf{F}$, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm d.

EXAMPLE 4.4

Two forces act on the rod shown in Fig. 4–15*a*. Determine the resultant moment they create about the flange at O. Express the result as a Cartesian vector.



(a)



(b)

SOLUTION

Position vectors are directed from point O to each force as shown in Fig. 4–15*b*. These vectors are

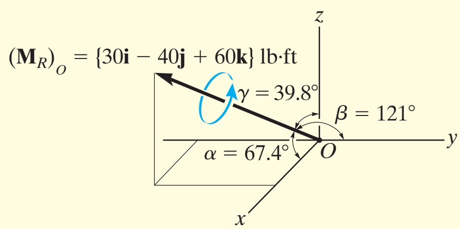
$$\mathbf{r}_A = \{5\mathbf{j}\}\text{ ft}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}\text{ ft}$$

The resultant moment about O is therefore

$$\begin{aligned}
(\mathbf{M}_R)_O &= \Sigma(\mathbf{r} \times \mathbf{F}) \\
&= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2 \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix} \\
&= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k} \\
&\quad + [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k} \\
&= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\text{ lb}\cdot\text{ft}
\end{aligned}$$

Ans.



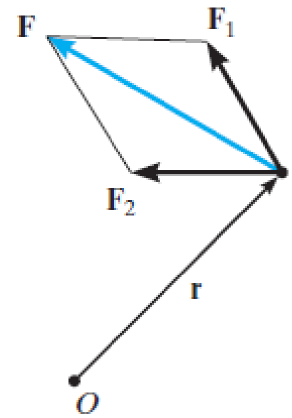
(c)

Fig. 4–15

NOTE: This result is shown in Fig. 4–15*c*. The coordinate direction angles were determined from the unit vector for $(\mathbf{M}_R)_O$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

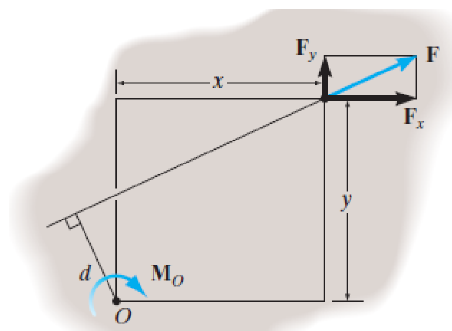
4.4 Principle of Moments

A concept often used in mechanics is the principle of moments, which is sometimes referred to as Varignon's theorem since it was orginally developed by the French mathematician Pierre Varignon (1654–1722). It states that the moment of a force about a point is equal to the sum of the moments of the components of the force about the point. This theorem can be proven easily using the vector cross product since the cross product obeys the distributive law. For example, consider the moments of the force F and two of its components about point O, Fig below. Since **F = F1 + F2** we have



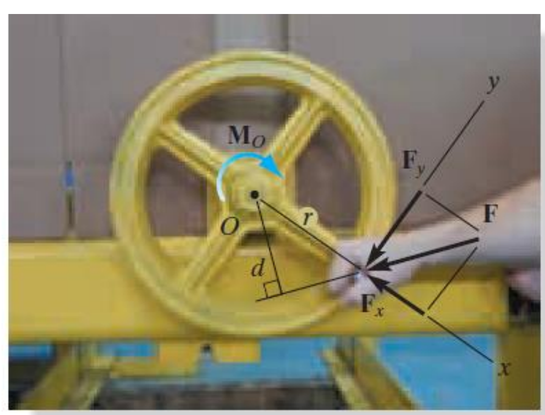
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

For two-dimensional problems, Fig, we can use the principle of moments by resolving the force into its rectangular components and then determine the moment using a scalar analysis. Thus,



$$\mathbf{M_O = F_x y - F_y x}$$

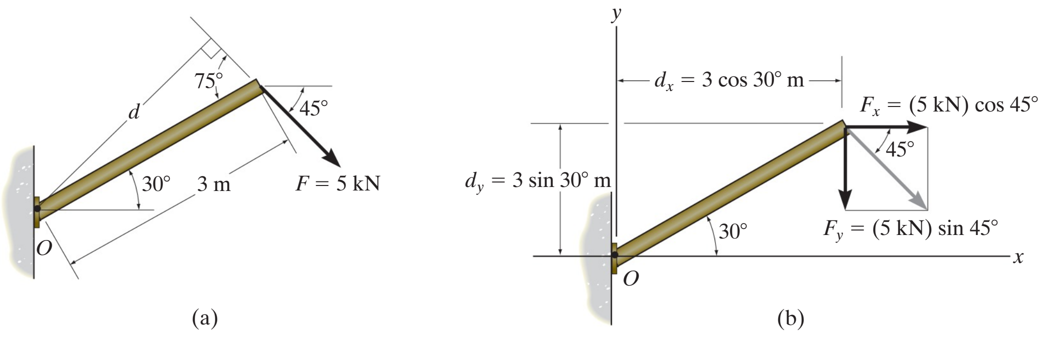
For real example : the moment of the force about point O is $\mathbf{M_O = Fd}$. But it is easier to find this moment using :



$$\mathbf{M_O = F_x(0) + F_y r = F_y r.}$$

EXAMPLE 4.5

Determine the moment of the force in Fig. 4–18*a* about point *O*.



SOLUTION I

The moment arm *d* in Fig. 4–18*a* can be found from trigonometry.

$$d = (3\text{ m})\sin 75^\circ = 2.898\text{ m}$$

Thus,

$$M_O = Fd = (5\text{ kN})(2.898\text{ m}) = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

SOLUTION II

The *x* and *y* components of the force are indicated in Fig. 4–18*b*. Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$\circlearrowleft + M_O = -F_x d_y - F_y d_x$$

$$= -(5\cos 45^\circ\text{ kN})(3\sin 30^\circ\text{ m}) - (5\sin 45^\circ\text{ kN})(3\cos 30^\circ\text{ m})$$

$$= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

SOLUTION III

The *x* and *y* axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 4–18*c*. Here $\mathbf{F}_x$ produces no moment about point *O* since its line of action passes through this point. Therefore,

$$\circlearrowleft + M_O = -F_y d_x$$

$$= -(5\sin 75^\circ\text{ kN})(3\text{ m})$$

$$= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

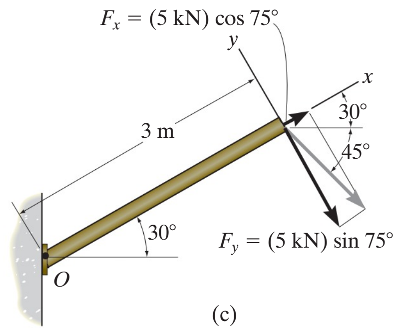


Fig. 4–18

Force **F** acts at the end of the angle bracket in Fig. 4–19*a*. Determine the moment of the force about point *O*.

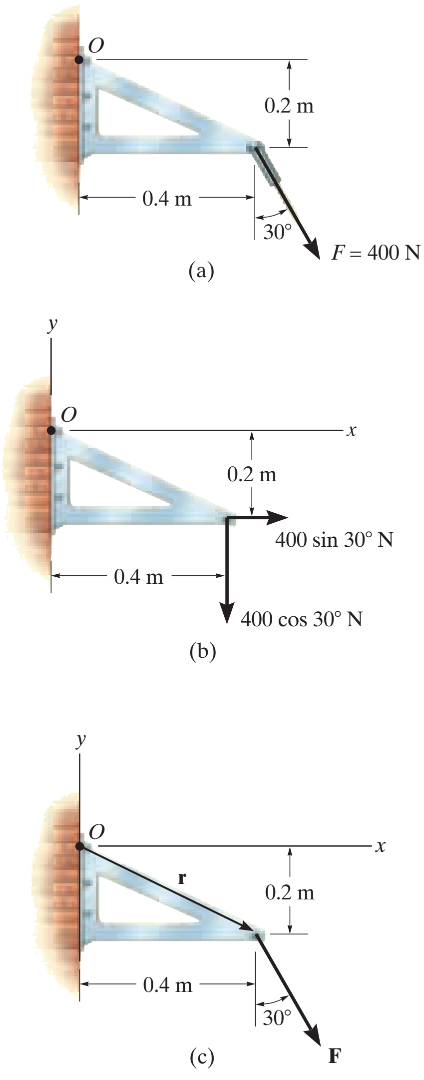


Fig. 4–19

SOLUTION I *(SCALAR ANALYSIS)*

The force is resolved into its *x* and *y* components, Fig. 4–19*b*, then

$$\circlearrowleft + M_O = 400 \sin 30^\circ \text{ N}(0.2 \text{ m}) - 400 \cos 30^\circ \text{ N}(0.4 \text{ m})$$

$$= -98.6 \text{ N}\cdot\text{m} = 98.6 \text{ N}\cdot\text{m} \circlearrowright$$

or

$$\mathbf{M}_O = \{-98.6\mathbf{k}\} \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

SOLUTION II *(VECTOR ANALYSIS)*

Using a Cartesian vector approach, the force and position vectors, Fig. 4–19*c*, are

$$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\} \text{ m}$$

$$\mathbf{F} = \{400 \sin 30^\circ \mathbf{i} - 400 \cos 30^\circ \mathbf{j}\} \text{ N}$$

$$= \{200.0\mathbf{i} - 346.4\mathbf{j}\} \text{ N}$$

The moment is therefore

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$

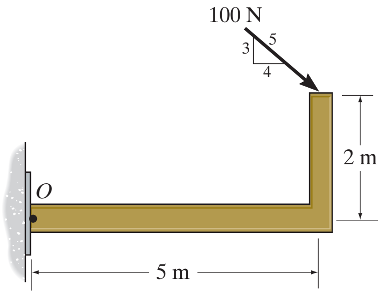
$$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$$

$$= \{-98.6\mathbf{k}\} \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: It is seen that the scalar analysis (Solution I) provides a more *convenient method* for analysis than Solution II since the direction of the moment and the moment arm for each component force are easy to establish. Hence, this method is generally recommended for solving problems displayed in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three-dimensional problems.

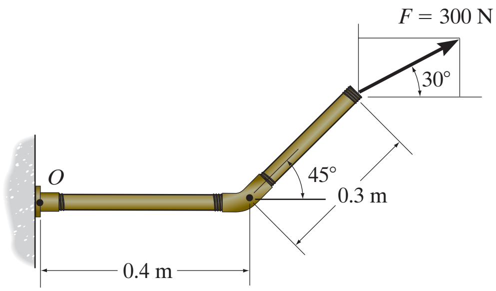
FUNDAMENTAL PROBLEMS

F4–1. Determine the moment of the force about point O.



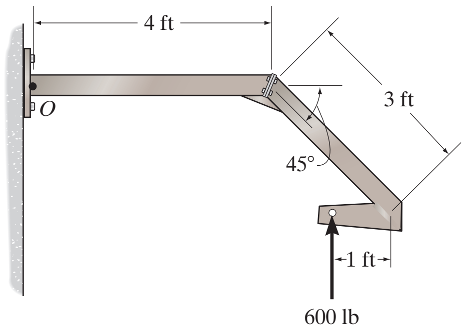
Prob. F4–1

F4–2. Determine the moment of the force about point O.



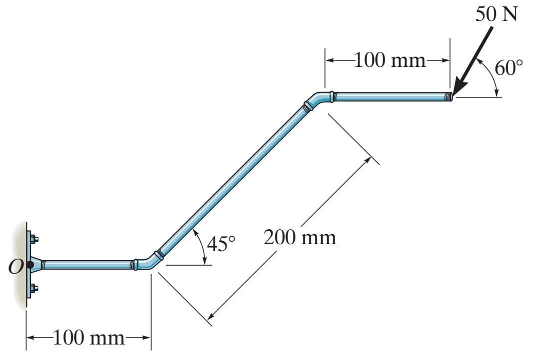
Prob. F4–2

F4–3. Determine the moment of the force about point O.



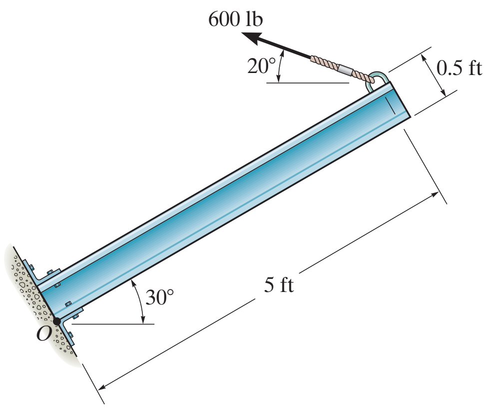
Prob. F4–3

F4–4. Determine the moment of the force about point O. Neglect the thickness of the member.



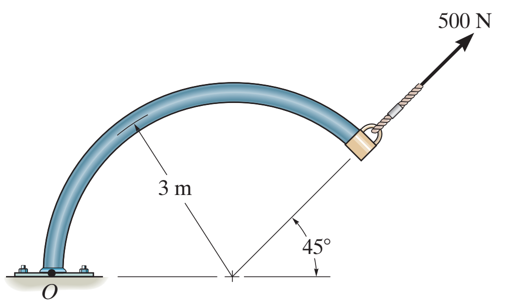
Prob. F4–4

F4–5. Determine the moment of the force about point O.



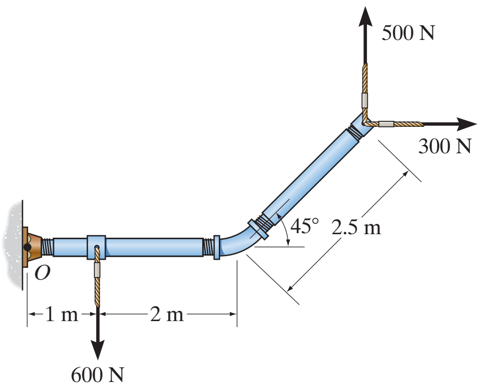
Prob. F4–5

F4–6. Determine the moment of the force about point O.



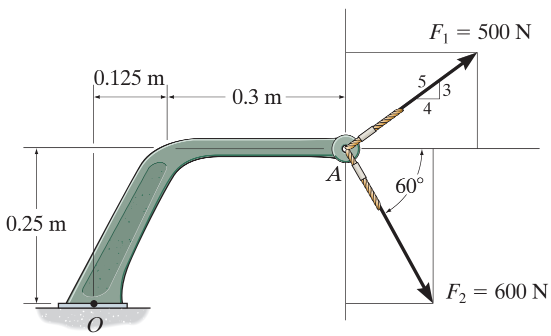
Prob. F4–6

F4–7. Determine the resultant moment produced by the forces about point O.



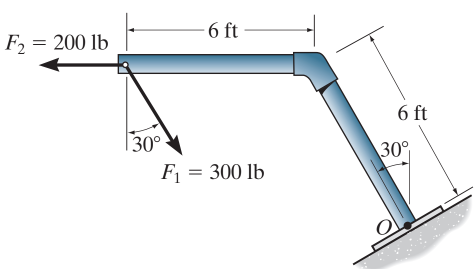
Prob. F4–7

F4–8. Determine the resultant moment produced by the forces about point O.



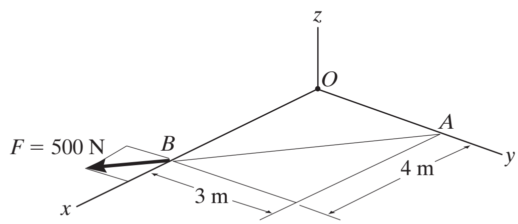
Prob. F4–8

F4–9. Determine the resultant moment produced by the forces about point O.



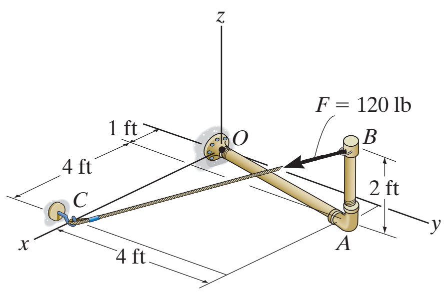
Prob. F4–9

F4–10. Determine the moment of force **F** about point O. Express the result as a Cartesian vector.



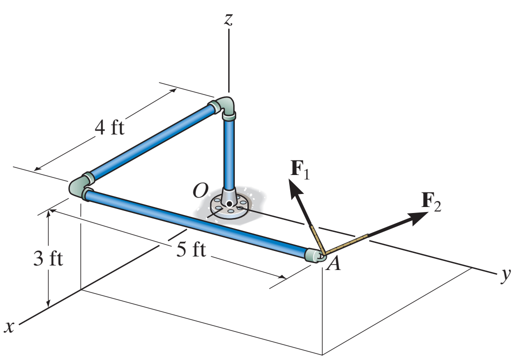
Prob. F4–10

F4–11. Determine the moment of force **F** about point O. Express the result as a Cartesian vector.



Prob. F4–11

F4–12. If the two forces $\mathbf{F}_1 = \{100\mathbf{i} - 120\mathbf{j} + 75\mathbf{k}\}$ lb and $\mathbf{F}_2 = \{-200\mathbf{i} + 250\mathbf{j} + 100\mathbf{k}\}$ lb act at A, determine the resultant moment produced by these forces about point O. Express the result as a Cartesian vector.



Prob. F4–12

Lecture 5

4.5 Moment of a Force about a Specified Axis

Sometimes, the moment produced by a force about a specified axis must be determined. For example, suppose the lug nut at O on the car tire in Fig. 4–20a needs to be loosened. The force applied to the wrench will create a tendency for the wrench and the nut to rotate about the moment axis passing through O; however, the nut can only rotate about the y axis. Therefore, to determine the turning effect, only the y component of the moment is needed, and the total moment produced is not important.

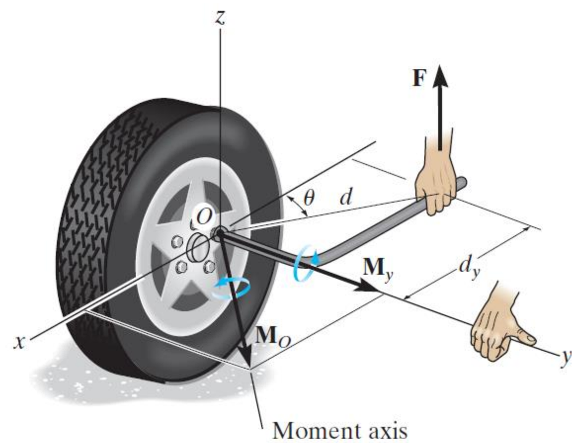
To determine this component, we can use either a **scalar** or **vector analysis**.

- **Scalar Analysis**: To use a scalar analysis in the case of the lug nut in Fig. 4–20a, the moment arm, or perpendicular distance from the axis to the line of action of the force, is

  $d_y = d \cos \theta$. Thus, the moment of **F** about the **y axis** is:

  $$M_y = F\, d_y = F(d \cos \theta)$$

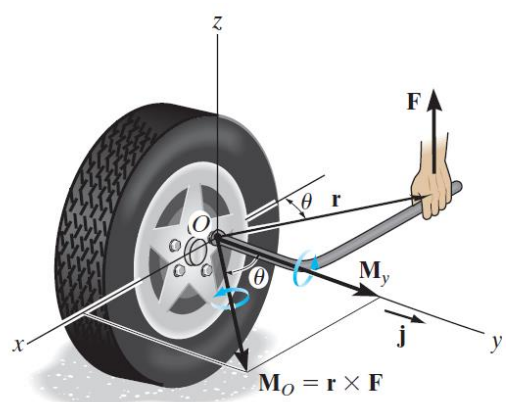
  According to the right-hand rule, My is directed along the positive y axis as shown in the figure.



- **Vector Analysis:** To find the moment of force **F** in Fig. about the y axis using a vector analysis:
  1. We must first determine the moment of the force about any point *O on the y axis* by applying:

     $$\mathbf{M_O = r \times F}$$

  2. The component My along the y axis is the **projection** of $\mathbf{M_O}$ onto the **y axis**.



It can be found using the **dot product** discussed in Chapter 2. so that

M_y = **j . M**$_O$ = **j . (r x F)**, where **j** is the **unit vector** for the y axis.

The **projection** of this moment onto the a axis is

$$M_a = \mathbf{u_a} \cdot (\mathbf{r} \times \mathbf{F})$$

This result can also be written in the form of a determinant, making it easier to memorize.

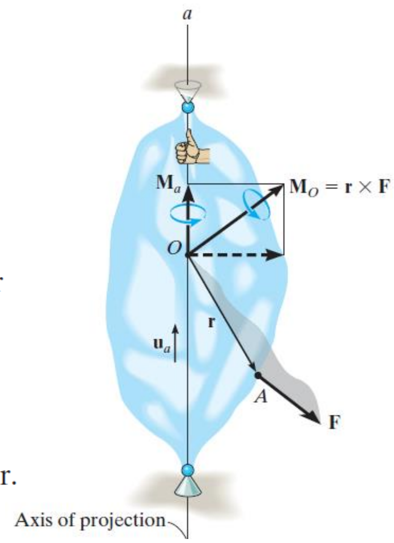
$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

where

$u_{a_x}, u_{a_y}, u_{a_z}$ represent the *x, y, z* components of the unit vector defining the direction of the *a* axis

r_x, r_y, r_z represent the *x, y, z* components of the position vector extended from *any point O* on the *a* axis to *any point A* on the line of action of the force

F_x, F_y, F_z represent the *x, y, z* components of the force vector.



Provided M_a is determined, we can then express **M**$_a$ as a **Cartesian vector**, namely,

$$\mathbf{M_a} = M_a \mathbf{u_a}$$

EXAMPLE 4.7

Determine the resultant moment of the three forces in Fig. 4–22 about the *x* axis, the *y* axis, and the *z* axis.

SOLUTION

A force that is *parallel* to a coordinate axis or has a line of action that passes through the axis does *not* produce any moment or tendency for turning about that axis. Therefore, defining the positive direction of the moment of a force according to the right-hand rule, as shown in the figure, we have

$$M_x = (60 \text{ lb})(2 \text{ ft}) + (50 \text{ lb})(2 \text{ ft}) + 0 = 220 \text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

$$M_y = 0 - (50 \text{ lb})(3 \text{ ft}) - (40 \text{ lb})(2 \text{ ft}) = -230 \text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

$$M_z = 0 + 0 - (40 \text{ lb})(2 \text{ ft}) = -80 \text{ lb}\cdot\text{ft} \quad \textit{Ans.}$$

The negative signs indicate that $\mathbf{M}_y$ and $\mathbf{M}_z$ act in the $-y$ and $-z$ directions, respectively.

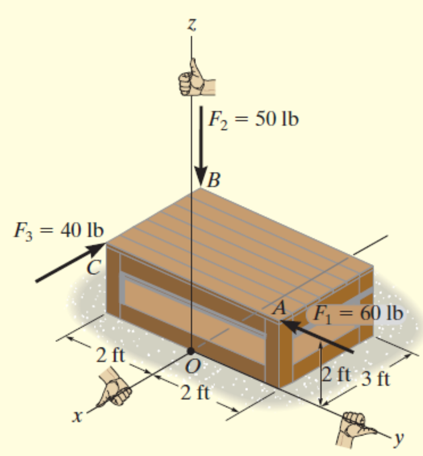


Fig. 4–22

EXAMPLE 4.8

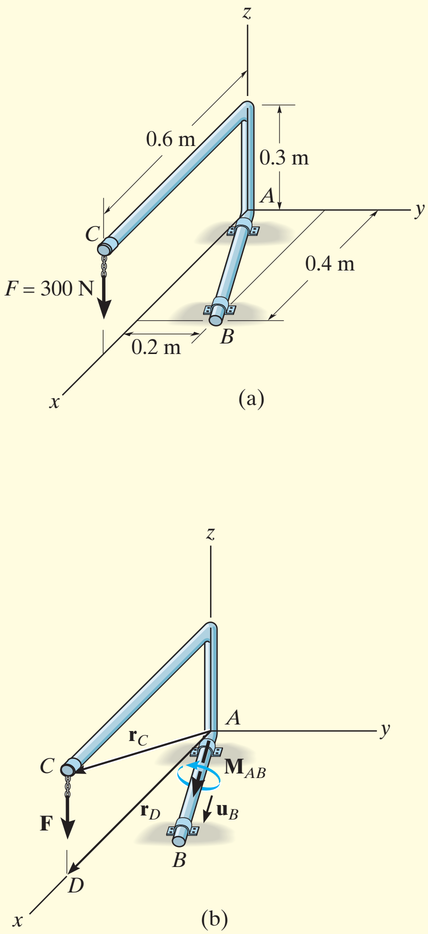


Fig. 4–23

Determine the moment $\mathbf{M}_{AB}$ produced by the force $\mathbf{F}$ in Fig. 4–23*a*, which tends to rotate the rod about the *AB* axis.

SOLUTION

A vector analysis using $M_{AB} = \mathbf{u}_B \cdot (\mathbf{r} \times \mathbf{F})$ will be considered for the solution rather than trying to find the moment arm or perpendicular distance from the line of action of $\mathbf{F}$ to the *AB* axis. Each of the terms in the equation will now be identified.

Unit vector $\mathbf{u}_B$ defines the direction of the *AB* axis of the rod, Fig. 4–23*b*, where

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{\{0.4\mathbf{i} + 0.2\mathbf{j}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (0.2\text{ m})^2}} = 0.8944\mathbf{i} + 0.4472\mathbf{j}$$

Vector $\mathbf{r}$ is directed from *any point* on the *AB* axis to *any point* on the line of action of the force. For example, position vectors $\mathbf{r}_C$ and $\mathbf{r}_D$ are suitable, Fig. 4–23*b*. (Although not shown, $\mathbf{r}_{BC}$ or $\mathbf{r}_{BD}$ can also be used.) For simplicity, we choose $\mathbf{r}_D$, where

$$\mathbf{r}_D = \{0.6\mathbf{i}\}\text{ m}$$

The force is

$$\mathbf{F} = \{-300\mathbf{k}\}\text{ N}$$

Substituting these vectors into the determinant form and expanding, we have

$$M_{AB} = \mathbf{u}_B \cdot (\mathbf{r}_D \times \mathbf{F}) = \begin{vmatrix} 0.8944 & 0.4472 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & -300 \end{vmatrix}$$

$$= 0.8944[0(-300) - 0(0)] - 0.4472[0.6(-300) - 0(0)] + 0[0.6(0) - 0(0)]$$

$$= 80.50\text{ N}\cdot\text{m}$$

This positive result indicates that the sense of $\mathbf{M}_{AB}$ is in the same direction as $\mathbf{u}_B$.

Expressing $\mathbf{M}_{AB}$ in Fig. 4–23*b* as a Cartesian vector yields

$$\mathbf{M}_{AB} = M_{AB}\mathbf{u}_B = (80.50\text{ N}\cdot\text{m})(0.8944\mathbf{i} + 0.4472\mathbf{j})$$

$$= \{72.0\mathbf{i} + 36.0\mathbf{j}\}\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: If axis *AB* is defined using a unit vector directed from *B* toward *A*, then in the above formulation $-\mathbf{u}_B$ would have to be used. This would lead to $M_{AB} = -80.50\text{ N}\cdot\text{m}$. Consequently, $\mathbf{M}_{AB} = M_{AB}(-\mathbf{u}_B)$, and the same result would be obtained.

EXAMPLE 4.9

Determine the magnitude of the moment of force **F** about segment *OA* of the pipe assembly in Fig. 4–24*a*.

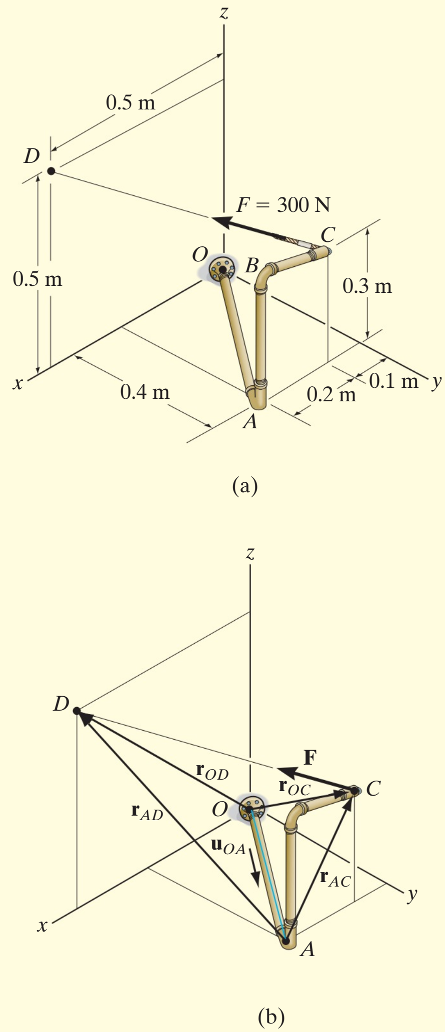


Fig. 4–24

SOLUTION

The moment of **F** about the *OA* axis is determined from $M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r} \times \mathbf{F})$, where **r** is a position vector extending from any point on the *OA* axis to any point on the line of action of **F**. As indicated in Fig. 4–24*b*, either $\mathbf{r}_{OD}$, $\mathbf{r}_{OC}$, $\mathbf{r}_{AD}$, or $\mathbf{r}_{AC}$ can be used; however, $\mathbf{r}_{OD}$ will be considered since it will simplify the calculation.

The unit vector $\mathbf{u}_{OA}$, which specifies the direction of the *OA* axis, is

$$\mathbf{u}_{OA} = \frac{\mathbf{r}_{OA}}{r_{OA}} = \frac{\{0.3\mathbf{i} + 0.4\mathbf{j}\}\text{ m}}{\sqrt{(0.3\text{ m})^2 + (0.4\text{ m})^2}} = 0.6\mathbf{i} + 0.8\mathbf{j}$$

and the position vector $\mathbf{r}_{OD}$ is

$$\mathbf{r}_{OD} = \{0.5\mathbf{i} + 0.5\mathbf{k}\}\text{ m}$$

The force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\left(\frac{\mathbf{r}_{CD}}{r_{CD}}\right)$$

$$= (300\text{ N})\left[\frac{\{0.4\mathbf{i} - 0.4\mathbf{j} + 0.2\mathbf{k}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (-0.4\text{ m})^2 + (0.2\text{ m})^2}}\right]$$

$$= \{200\mathbf{i} - 200\mathbf{j} + 100\mathbf{k}\}\text{ N}$$

Therefore,

$$M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r}_{OD} \times \mathbf{F})$$

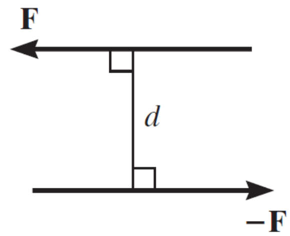
$$= \begin{vmatrix} 0.6 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 200 & -200 & 100 \end{vmatrix}$$

$$= 0.6[0(100) - (0.5)(-200)] - 0.8[0.5(100) - (0.5)(200)] + 0$$

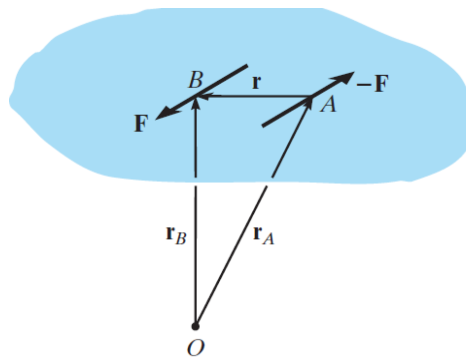
$$= 100\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

4.6 Moment of a Couple

A couple is defined as two parallel forces that have the same magnitude, but opposite directions, and are separated by a perpendicular distance **d**, Fig.



The moment produced by a couple is called a **couple moment**. We can determine its value by finding the sum of the moments of both couple forces about any arbitrary point. For example, in Fig, position vectors $\mathbf{r_A}$ and $\mathbf{r_B}$ are directed from point **O** to points **A** and **B** lying on **the line of action** of **-F** and **F**. The couple moment determined about O is therefore



$$\mathbf{M} = r_B \times F + r_A \times -F = (\mathbf{r_B} - \mathbf{r_A}) \times \mathbf{F}$$

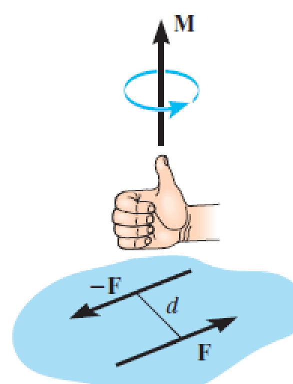
$$\mathbf{M = r \times F}$$

This result indicates that a couple moment is a free vector, i.e., it can act at any point since M depends only upon the position vector r directed between the forces and not the position vectors r_A and r_B.

Scalar Formulation: The moment of a couple, **M**, Fig, is defined as having a **magnitude** of

$$M = Fd$$

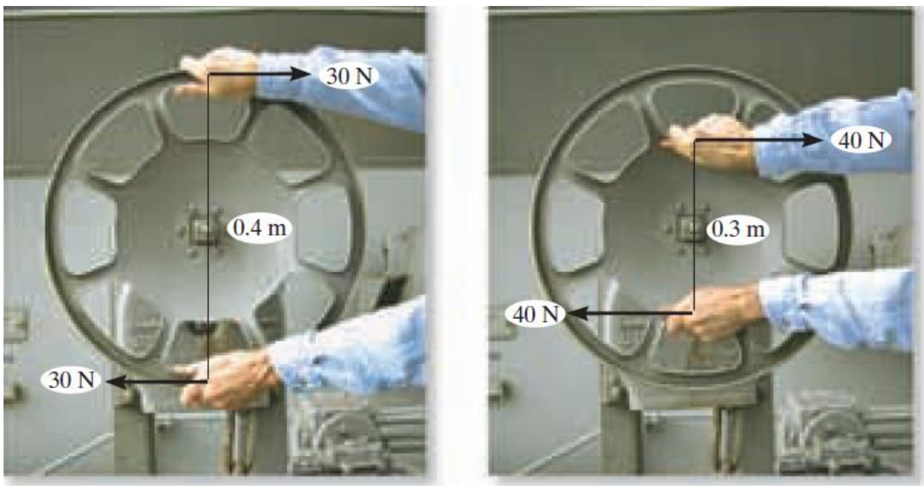
where F is the magnitude of one of the forces and d is the perpendicular distance or moment arm between the forces. The direction and sense of the couple moment are determined by the right-hand rule.



Vector Formulation: The moment of a couple can also be expressed by the vector cross product using.

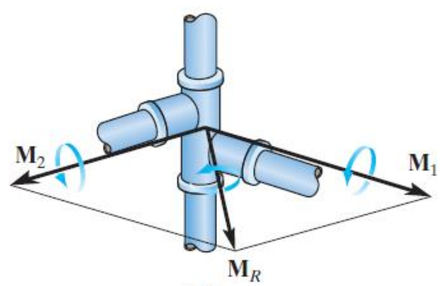
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Equivalent Couples: Two couples produce a moment with the *same magnitude and direction*, then these two couples are *equivalent*. For example, the two couples shown in Fig. are *equivalent* because each couple moment has a magnitude of M = **30 N(0.4 m) = 40 N(0.3 m) = 12 N . m**, and each is directed into the plane of the page.



Resultant Couple Moment.

Since couple moments are vectors, their resultant can be determined by vector addition. For example, consider the couple moments $\mathbf{M}_1$ and $\mathbf{M}_2$ acting on the pipe. the resultant couple moment, $\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2$ as shown in Fig.



(b)

we may generalize this concept and write the vector resultant as :

$$\mathbf{M}_R = \sum (\mathbf{r} \times \mathbf{F})$$

EXAMPLE 4.10

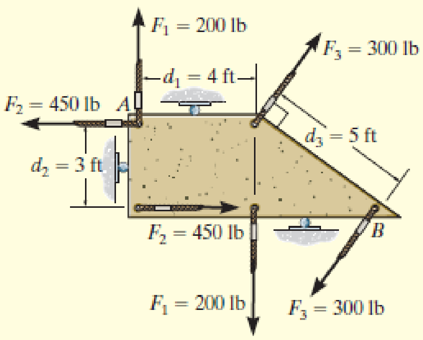


Fig. 4–30

Determine the resultant couple moment of the three couples acting on the plate in Fig. 4–30.

SOLUTION

As shown the perpendicular distances between each pair of couple forces are $d_1 = 4$ ft, $d_2 = 3$ ft, and $d_3 = 5$ ft. Considering counterclockwise couple moments as positive, we have

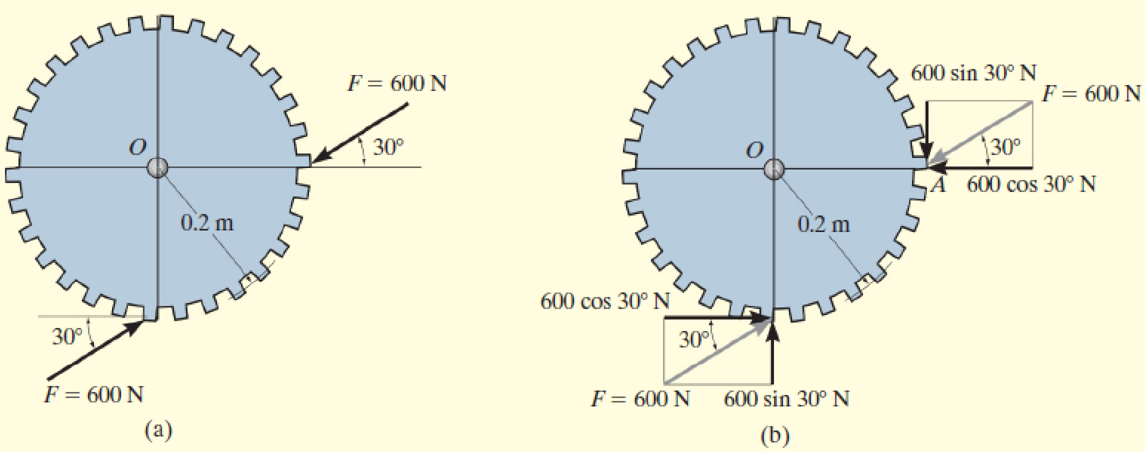
$$\circlearrowleft + M_R = \Sigma M; \; M_R = -F_1 d_1 + F_2 d_2 - F_3 d_3$$

$$= -(200 \text{ lb})(4 \text{ ft}) + (450 \text{ lb})(3 \text{ ft}) - (300 \text{ lb})(5 \text{ ft})$$

$$= -950 \text{ lb} \cdot \text{ft} = 950 \text{ lb} \cdot \text{ft} \circlearrowright \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{M}_R$ has a clockwise rotational sense.

Example 4.11 :Determine the magnitude and direction of the couple moment acting on the gear in Fig. a.



SOLUTION

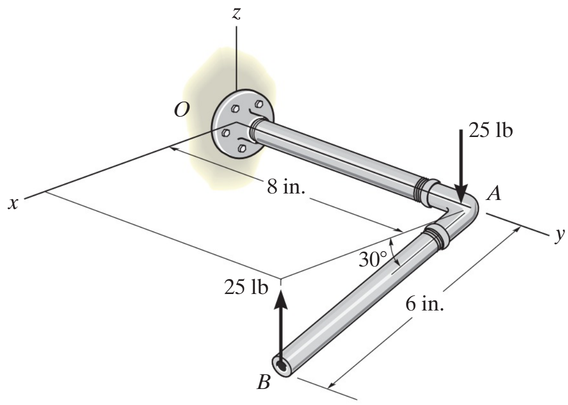
The easiest solution requires resolving each force into its components as shown in Fig. 4–31*b*. The couple moment can be determined by summing the moments of these force components about any point, for example, the center *O* of the gear or point *A*. If we consider counterclockwise moments as positive, we have

$$\circlearrowleft + M = \Sigma M_O; \; M = (600 \cos 30° \text{ N})(0.2 \text{ m}) - (600 \sin 30° \text{ N})(0.2 \text{ m})$$

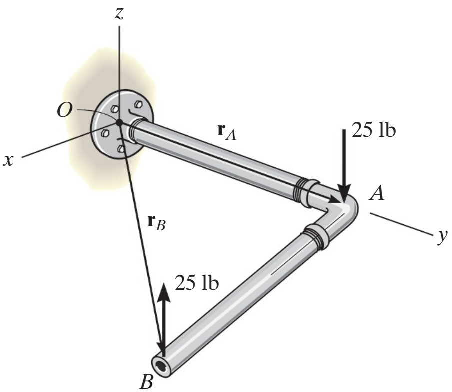
$$= 43.9 \text{ N} \cdot \text{m} \circlearrowleft \qquad \textit{Ans.}$$

EXAMPLE 4.12

Determine the couple moment acting on the pipe shown in Fig. 4–32*a*. Segment AB is directed 30° below the x–y plane.



(a)

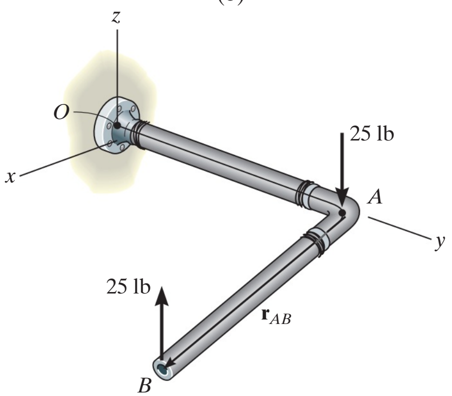


(b)

SOLUTION I *(VECTOR ANALYSIS)*

The moment of the two couple forces can be found about *any point*. If point O is considered, Fig. 4–32*b*, we have

$$\mathbf{M} = \mathbf{r}_A \times (-25\mathbf{k}) + \mathbf{r}_B \times (25\mathbf{k})$$
$$= (8\mathbf{j}) \times (-25\mathbf{k}) + (6 \cos 30°\mathbf{i} + 8\mathbf{j} - 6 \sin 30°\mathbf{k}) \times (25\mathbf{k})$$
$$= -200\mathbf{i} - 129.9\mathbf{j} + 200\mathbf{i}$$
$$= \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.}$$ *Ans.*



(c)

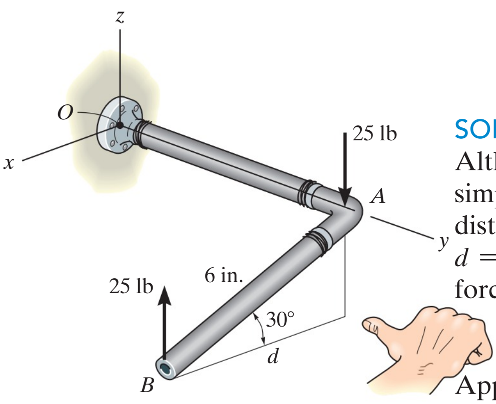
It is *easier* to take moments of the couple forces about a point lying on the line of action of one of the forces, e.g., point A, Fig. 4–32*c*. In this case the moment of the force at A is zero, so that

$$\mathbf{M} = \mathbf{r}_{AB} \times (25\mathbf{k})$$
$$= (6 \cos 30°\mathbf{i} - 6 \sin 30°\mathbf{k}) \times (25\mathbf{k})$$
$$= \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.}$$ *Ans.*

SOLUTION II *(SCALAR ANALYSIS)*

Although this problem is shown in three dimensions, the geometry is simple enough to use the scalar equation $M = Fd$. The perpendicular distance between the lines of action of the couple forces is $d = 6 \cos 30° = 5.196$ in., Fig. 4–32*d*. Hence, taking moments of the forces about either point A or point B yields

$$M = Fd = 25 \text{ lb } (5.196 \text{ in.}) = 129.9 \text{ lb} \cdot \text{in.}$$



(d)

Fig. 4–32

Applying the right-hand rule, $\mathbf{M}$ acts in the $-\mathbf{j}$ direction. Thus,

$$\mathbf{M} = \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.}$$ *Ans.*

EXAMPLE 4.13

Replace the two couples acting on the pipe column in Fig. 4–33*a* by a resultant couple moment.

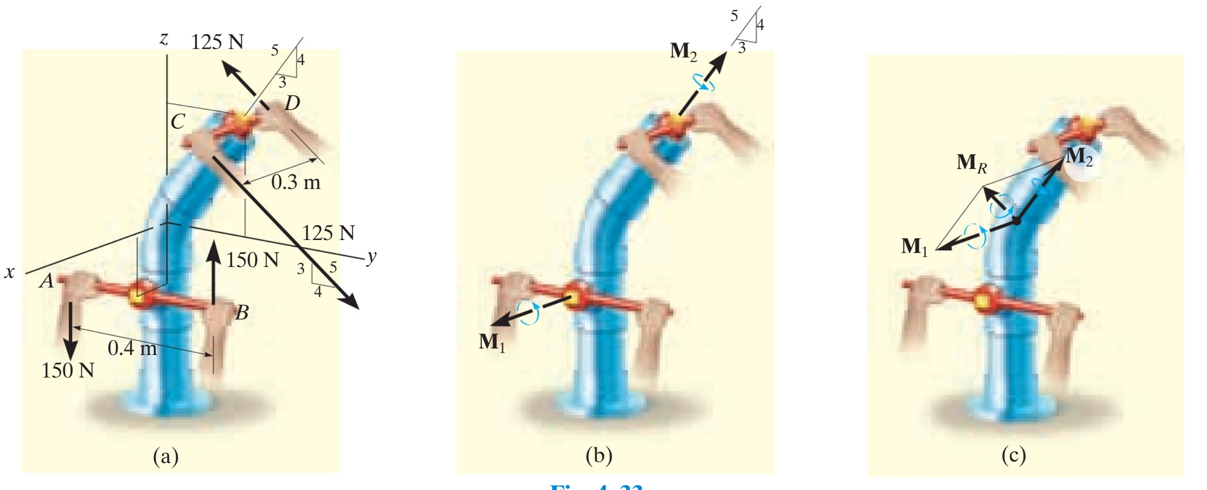


Fig. 4–33

SOLUTION *(VECTOR ANALYSIS)*

The couple moment $\mathbf{M}_1$, developed by the forces at A and B, can easily be determined from a scalar formulation.

$$M_1 = Fd = 150\text{ N}(0.4\text{ m}) = 60\text{ N}\cdot\text{m}$$

By the right-hand rule, $\mathbf{M}_1$ acts in the $+\mathbf{i}$ direction, Fig. 4–33*b*. Hence,

$$\mathbf{M}_1 = \{60\mathbf{i}\}\text{ N}\cdot\text{m}$$

Vector analysis will be used to determine $\mathbf{M}_2$, caused by forces at C and D. If moments are calculated about point D, Fig. 4–33*a*, $\mathbf{M}_2 = \mathbf{r}_{DC} \times \mathbf{F}_C$, then

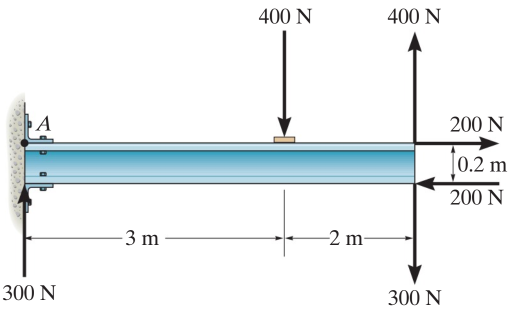
$$\begin{aligned}\mathbf{M}_2 = \mathbf{r}_{DC} \times \mathbf{F}_C &= (0.3\mathbf{i}) \times \left[125\left(\tfrac{4}{5}\right)\mathbf{j} - 125\left(\tfrac{3}{5}\right)\mathbf{k}\right] \\ &= (0.3\mathbf{i}) \times [100\mathbf{j} - 75\mathbf{k}] = 30(\mathbf{i} \times \mathbf{j}) - 22.5(\mathbf{i} \times \mathbf{k}) \\ &= \{22.5\mathbf{j} + 30\mathbf{k}\}\text{ N}\cdot\text{m}\end{aligned}$$

Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are free vectors, they may be moved to some arbitrary point and added vectorially, Fig. 4–33*c*. The resultant couple moment becomes

$$\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2 = \{60\mathbf{i} + 22.5\mathbf{j} + 30\mathbf{k}\}\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

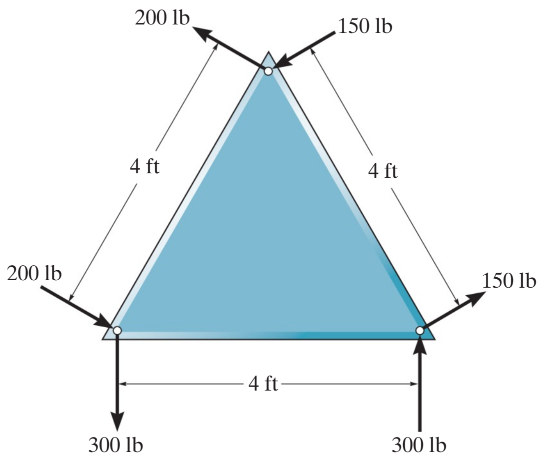
FUNDAMENTAL PROBLEMS

F4–19. Determine the resultant couple moment acting on the beam.



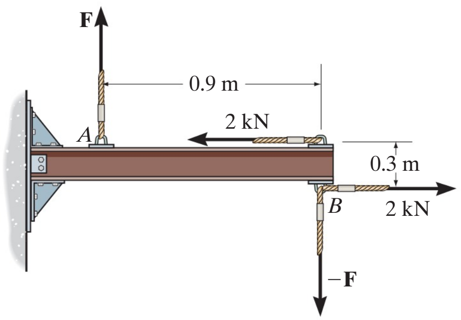
Prob. F4–19

F4–20. Determine the resultant couple moment acting on the triangular plate.



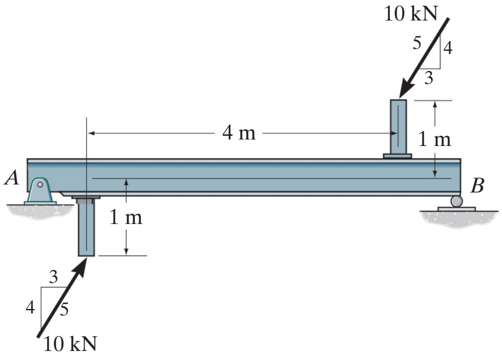
Prob. F4–20

F4–21. Determine the magnitude of **F** so that the resultant couple moment acting on the beam is 1.5 kN · m clockwise.



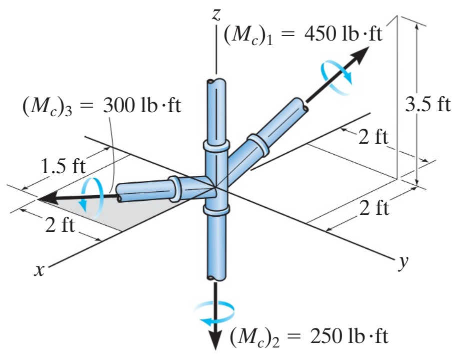
Prob. F4–21

F4–22. Determine the couple moment acting on the beam.



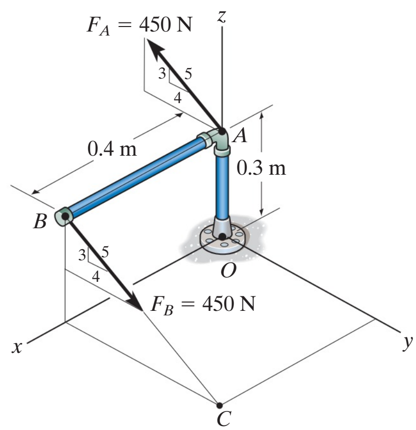
Prob. F4–22

F4–23. Determine the resultant couple moment acting on the pipe assembly.



Prob. F4–23

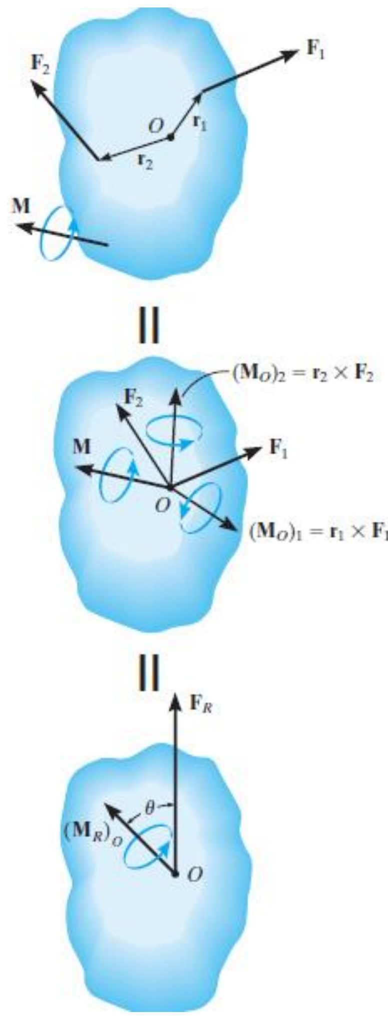
F4–24. Determine the couple moment acting on the pipe assembly and express the result as a Cartesian vector.



Prob. F4–24

4.7 Simplification of a Force and Couple System

Sometimes it is convenient to reduce a system of forces and couple moments acting on a body to a simpler form by replacing it with an equivalent system, consisting of a single resultant force acting at a specific point and a resultant couple moment. A system is equivalent if the external effects it produces on a body are the same as those caused by the original force and couple moment system.



System of Forces and Couple Moments. Using the abovemethod, a system of several forces and couple moments acting on a body can be reduced to an equivalent single resultant force acting at a point O. For example, in Fig.

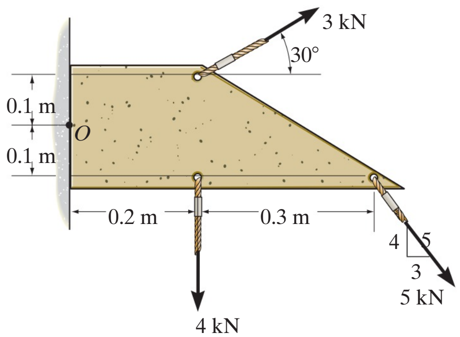
$$\mathbf{F}_R = \Sigma \mathbf{F}$$
$$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O + \Sigma \mathbf{M}$$

If the force system lies in the x–y plane and any couple moments are perpendicular to this plane, then the above equations reduce to the following three scalar equations.

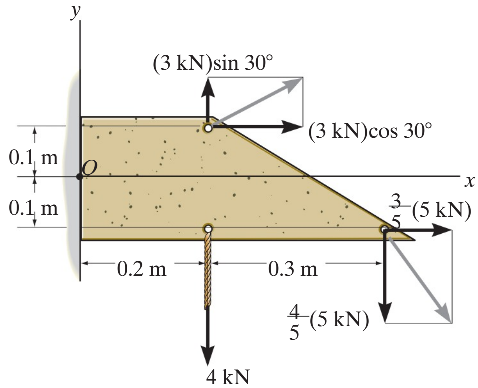
$$(F_R)_x = \Sigma F_x$$
$$(F_R)_y = \Sigma F_y$$
$$(M_R)_O = \Sigma M_O + \Sigma M$$

EXAMPLE 4.14

Replace the force and couple system shown in Fig. 4–37*a* by an equivalent resultant force and couple moment acting at point *O*.



(a)



(b)

SOLUTION

Force Summation. The 3 kN and 5 kN forces are resolved into their *x* and *y* components as shown in Fig. 4–37*b*. We have

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = (3\text{ kN})\cos 30° + \left(\tfrac{3}{5}\right)(5\text{ kN}) = 5.598\text{ kN} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = (3\text{ kN})\sin 30° - \left(\tfrac{4}{5}\right)(5\text{ kN}) - 4\text{ kN} = -6.50\text{ kN} = 6.50\text{ kN}\downarrow$$

Using the Pythagorean theorem, Fig. 4–37*c*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2} = \sqrt{(5.598\text{ kN})^2 + (6.50\text{ kN})^2} = 8.58\text{ kN} \qquad \textit{Ans.}$$

Its direction θ is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{6.50\text{ kN}}{5.598\text{ kN}}\right) = 49.3° \qquad \textit{Ans.}$$

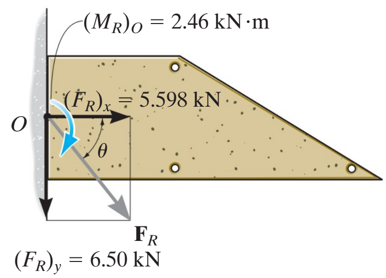
Moment Summation. The moments of 3 kN and 5 kN about point *O* will be determined using their *x* and *y* components. Referring to Fig. 4–37*b*, we have

$$\circlearrowleft + (M_R)_O = \Sigma M_O;$$

$$(M_R)_O = (3\text{ kN})\sin 30°(0.2\text{ m}) - (3\text{ kN})\cos 30°(0.1\text{ m}) + \left(\tfrac{3}{5}\right)(5\text{ kN})\,(0.1\text{ m}) - \left(\tfrac{4}{5}\right)(5\text{ kN})\,(0.5\text{ m}) - (4\text{ kN})(0.2\text{ m})$$

$$= -2.46\text{ kN}\cdot\text{m} = 2.46\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

This clockwise moment is shown in Fig. 4–37*c*.



(c)

Fig. 4–37

NOTE: Realize that the resultant force and couple moment in Fig. 4–37*c* will produce the same external effects or reactions at the supports as those produced by the force system, Fig. 4–37*a*.

EXAMPLE 4.15

Replace the force and couple system acting on the member in Fig. 4–38*a* by an equivalent resultant force and couple moment acting at point *O*.

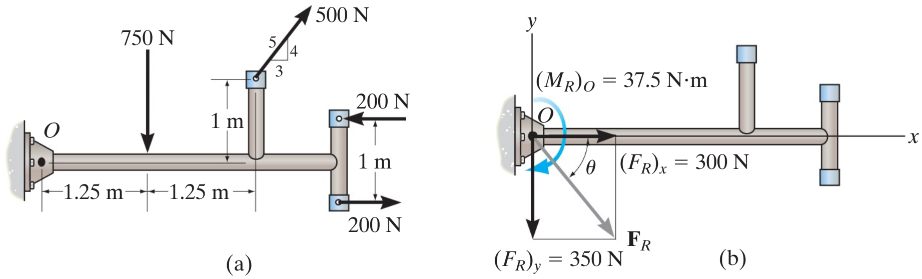


Fig. 4–38

SOLUTION

Force Summation. Since the couple forces of 200 N are equal but opposite, they produce a zero resultant force, and so it is not necessary to consider them in the force summation. The 500-N force is resolved into its *x* and *y* components, thus,

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \quad (F_R)_x = \left(\tfrac{3}{5}\right)(500\text{ N}) = 300\text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y = (500\text{ N})\left(\tfrac{4}{5}\right) - 750\text{ N} = -350\text{ N} = 350\text{ N}\downarrow$$

From Fig. 4–15*b*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

$$= \sqrt{(300\text{ N})^2 + (350\text{ N})^2} = 461\text{ N} \qquad \textit{Ans.}$$

And the angle θ is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{350\text{ N}}{300\text{ N}}\right) = 49.4^\circ \qquad \textit{Ans.}$$

Moment Summation. Since the couple moment is a free vector, it can act at any point on the member. Referring to Fig. 4–38*a*, we have

$$\circlearrowleft + (M_R)_O = \Sigma M_O + \Sigma M$$

$$(M_R)_O = (500\text{ N})\left(\tfrac{4}{5}\right)(2.5\text{ m}) - (500\text{ N})\left(\tfrac{3}{5}\right)(1\text{ m})$$

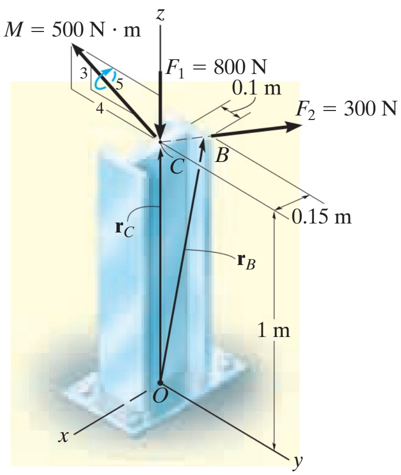
$$- (750\text{ N})(1.25\text{ m}) + 200\text{ N}\cdot\text{m}$$

$$= -37.5\text{ N}\cdot\text{m} = 37.5\text{ N}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

This clockwise moment is shown in Fig. 4–38*b*.

EXAMPLE 4.16

The structural member is subjected to a couple moment **M** and forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 4–39*a*. Replace this system by an equivalent resultant force and couple moment acting at its base, point *O*.



(a)

SOLUTION *(VECTOR ANALYSIS)*

The three-dimensional aspects of the problem can be simplified by using a Cartesian vector analysis. Expressing the forces and couple moment as Cartesian vectors, we have

$$\mathbf{F}_1 = \{-800\mathbf{k}\}\text{ N}$$

$$\mathbf{F}_2 = (300\text{ N})\mathbf{u}_{CB}$$

$$= (300\text{ N})\left(\frac{\mathbf{r}_{CB}}{r_{CB}}\right)$$

$$= 300\text{ N}\left[\frac{\{-0.15\mathbf{i} + 0.1\mathbf{j}\}\text{ m}}{\sqrt{(-0.15\text{ m})^2 + (0.1\text{ m})^2}}\right] = \{-249.6\mathbf{i} + 166.4\mathbf{j}\}\text{ N}$$

$$\mathbf{M} = -500\left(\tfrac{4}{5}\right)\mathbf{j} + 500\left(\tfrac{3}{5}\right)\mathbf{k} = \{-400\mathbf{j} + 300\mathbf{k}\}\text{ N}\cdot\text{m}$$

Force Summation.

$$\mathbf{F}_R = \Sigma\mathbf{F}; \qquad \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = -800\mathbf{k} - 249.6\mathbf{i} + 166.4\mathbf{j}$$

$$= \{-250\mathbf{i} + 166\mathbf{j} - 800\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

Moment Summation.

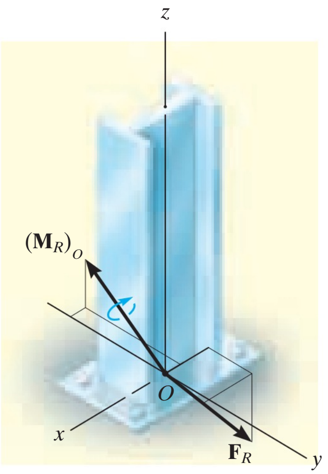
$$(\mathbf{M}_R)_o = \Sigma\mathbf{M} + \Sigma\mathbf{M}_O$$

$$(\mathbf{M}_R)_o = \mathbf{M} + \mathbf{r}_C \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$(\mathbf{M}_R)_o = (-400\mathbf{j} + 300\mathbf{k}) + (1\mathbf{k}) \times (-800\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix}$$

$$= (-400\mathbf{j} + 300\mathbf{k}) + (0) + (-166.4\mathbf{i} - 249.6\mathbf{j})$$

$$= \{-166\mathbf{i} - 650\mathbf{j} + 300\mathbf{k}\}\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$



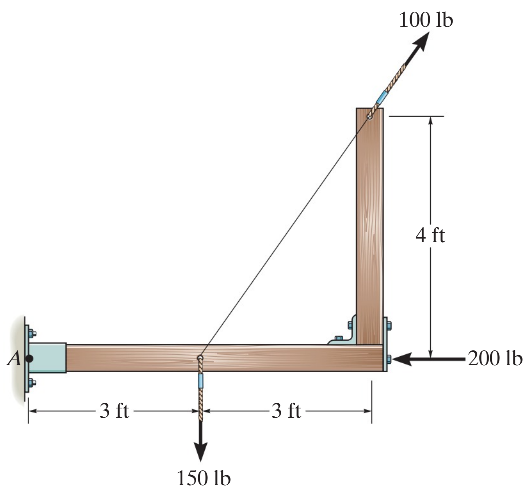
(b)

Fig. 4–39

The results are shown in Fig. 4–39*b*.

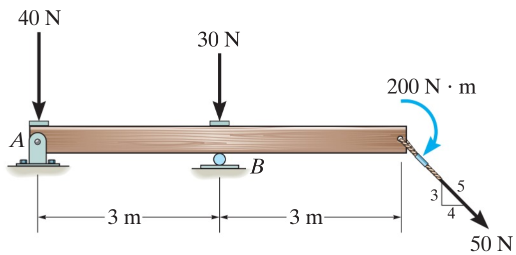
FUNDAMENTAL PROBLEMS

F4–25. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



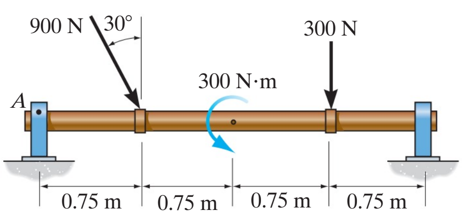
Prob. F4–25

F4–26. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



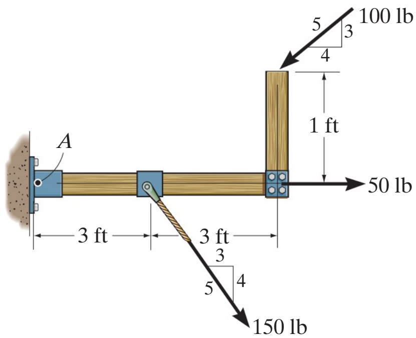
Prob. F4–26

F4–27. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



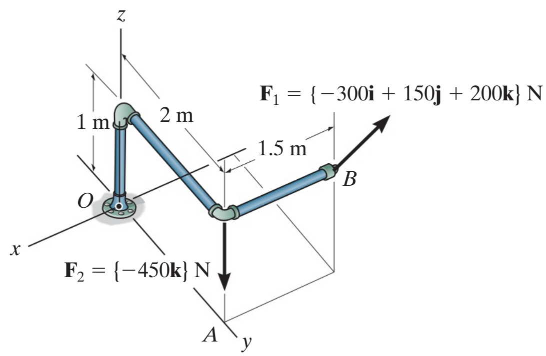
Prob. F4–27

F4–28. Replace the loading system by an equivalent resultant force and couple moment acting at point A.



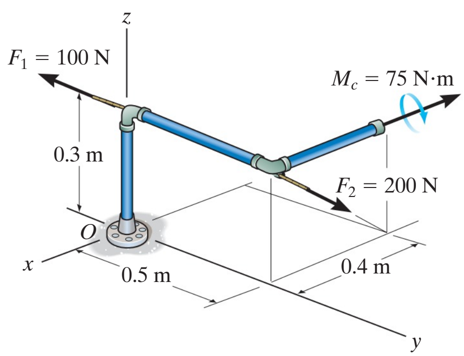
Prob. F4–28

F4–29. Replace the loading system by an equivalent resultant force and couple moment acting at point O.



Prob. F4–29

F4–30. Replace the loading system by an equivalent resultant force and couple moment acting at point O.



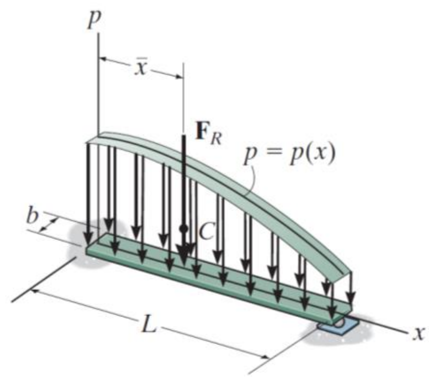
Prob. F4–30

Lecture 6

4.8 Reduction of a Simple Distributed Loading

Sometimes, a body may be subjected to a loading that is distributed over its surface. For example, the pressure of the wind on the face of a sign, the pressure of water within a tank, or the weight of sand on the floor of a storage container, are all **distributed loadings**.

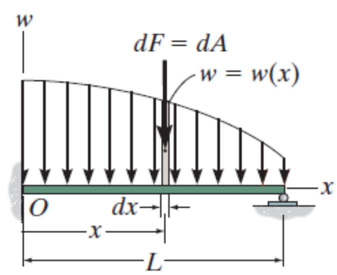
Loading Along a Single Axis. The most common type of distributed loading encountered in engineering practice can berepresented along a single axis. For example, consider the beam (or plate) in Fig. that has a constant width and is subjected to a pressure loading that varies only along the x axis.



we multiply the loading function by the width b m of the beam, so that

$$w(x) = p(x)b \text{ N/m},$$

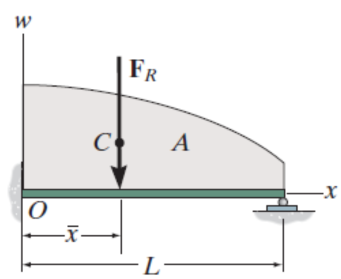
Magnitude of Resultant Force. The magnitude of $\mathbf{F_R}$ is equivalent to the sum of all the forces in the system.



(b)

$$+\downarrow F_R = \Sigma F; \quad F_R = \int_L w(x)\,dx = \int_A dA = A$$

Therefore, the magnitude of the resultant force is equal to the area A under the loading diagram



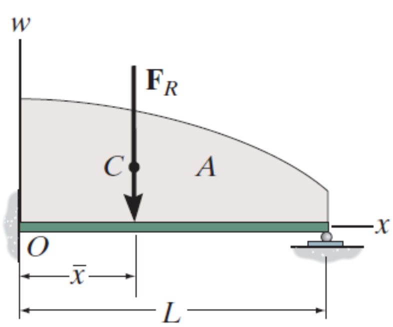
Location of Resultant Force. Applying ($M_{RO} = \sum M_O$), the location x of the line of action of $\mathbf{F}_R$ can be determined by equatingthe moments of the force resultant and the parallel force distribution about point O (the y axis).

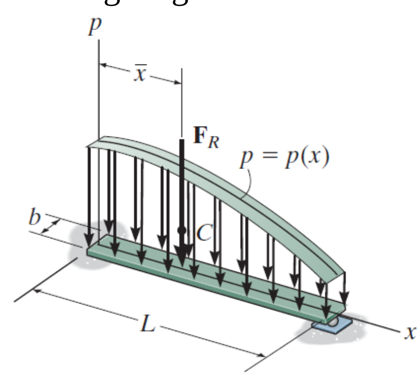
$$\circlearrowleft + (M_R)_O = \Sigma M_O; \qquad -\bar{x}F_R = -\int_L xw(x)\,dx$$

Solving for $\bar{x}$, using Eq. 4–19, we have

$$\bar{x} = \frac{\int_L xw(x)\,dx}{\int_L w(x)\,dx} = \frac{\int_A x\,dA}{\int_A dA}$$

This coordinate $\bar{x}$, locates the geometric center or centroid of the area under the distributed loading. In other words, the resultant force has a line of action which passes through the centroid C (geometric center) of the area under the loading diagram.





Important Points

- Coplanar distributed loadings are defined by using a loading function $w = w(x)$ that indicates the intensity of the loading along the length of a member. This intensity is measured in N/m or lb/ft.
- The external effects caused by a coplanar distributed load acting on a body can be represented by a single resultant force.
- This resultant force is equivalent to the *area* under the loading diagram, and has a line of action that passes through the *centroid* or geometric center of this area.

EXAMPLE 4.17

Determine the magnitude and location of the equivalent resultant force acting on the shaft in Fig. 4–49*a*.

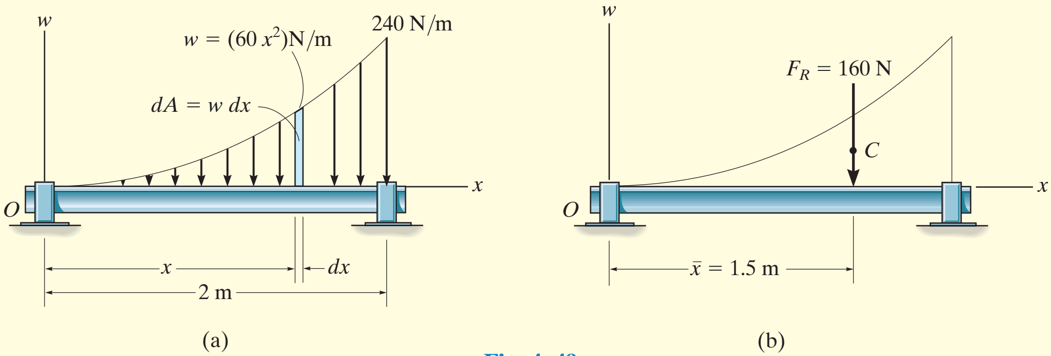


Fig. 4–49

SOLUTION

Since $w = w(x)$ is given, this problem will be solved by integration.

The differential element has an area $dA = w\,dx = 60x^2\,dx$. Applying Eq. 4–19,

$$+\downarrow F_R = \Sigma F;$$

$$F_R = \int_A dA = \int_0^{2\text{ m}} 60x^2\,dx = 60\left(\frac{x^3}{3}\right)\Bigg|_0^{2\text{ m}} = 60\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= 160 \text{ N}$$ *Ans.*

The location $\bar{x}$ of $\mathbf{F}_R$ measured from O, Fig. 4–49*b*, is determined from Eq. 4–20.

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^{2\text{ m}} x(60x^2)\,dx}{160\text{ N}} = \frac{60\left(\frac{x^4}{4}\right)\Big|_0^{2\text{ m}}}{160\text{ N}} = \frac{60\left(\frac{2^4}{4} - \frac{0^4}{4}\right)}{160\text{ N}}$$

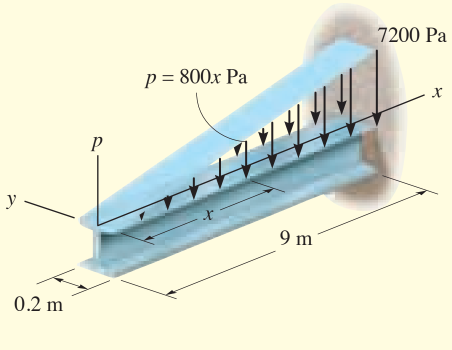
$$= 1.5 \text{ m}$$ *Ans.*

NOTE: These results can be checked by using the table on the inside back cover, where it is shown that the formula for an exparabolic area of length a, height b, and shape shown in Fig. 4–49*a*, is

$$A = \frac{ab}{3} = \frac{2\text{ m}(240\text{ N/m})}{3} = 160\text{ N and } \bar{x} = \frac{3}{4}a = \frac{3}{4}(2\text{ m}) = 1.5\text{ m}$$

EXAMPLE 4.18

A distributed loading of $p = (800x)$ Pa acts over the top surface of the beam shown in Fig. 4–50*a*. Determine the magnitude and location of the equivalent resultant force.

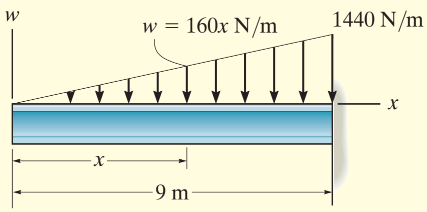


(a)

SOLUTION

Since the loading intensity is uniform along the width of the beam (the *y* axis), the loading can be viewed in two dimensions as shown in Fig. 4–50*b*. Here

$$w = (800x \text{ N/m}^2)(0.2 \text{ m})$$
$$= (160x) \text{ N/m}$$



(b)

At $x = 9$ m, note that $w = 1440$ N/m. Although we may again apply Eqs. 4–19 and 4–20 as in the previous example, it is simpler to use the table on the inside back cover.

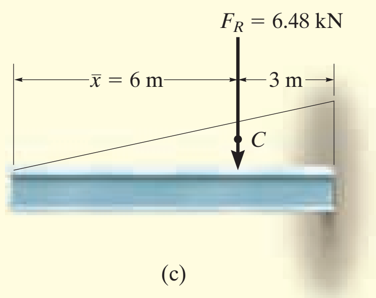
The magnitude of the resultant force is equivalent to the area of the triangle.

$$F_R = \tfrac{1}{2}(9 \text{ m})(1440 \text{ N/m}) = 6480 \text{ N} = 6.48 \text{ kN} \qquad \textit{Ans.}$$

The line of action of $\mathbf{F}_R$ passes through the *centroid C* of this triangle. Hence,

$$\bar{x} = 9 \text{ m} - \tfrac{1}{3}(9 \text{ m}) = 6 \text{ m} \qquad \textit{Ans.}$$

The results are shown in Fig. 4–50*c*.



(c)

NOTE: We may also view the resultant $\mathbf{F}_R$ as *acting* through the *centroid* of the *volume* of the loading diagram $p = p(x)$ in Fig. 4–50*a*. Hence $\mathbf{F}_R$ intersects the *x–y* plane at the point (6 m, 0). Furthermore, the magnitude of $\mathbf{F}_R$ is equal to the volume under the loading diagram; i.e.,

$$F_R = V = \tfrac{1}{2}(7200 \text{ N/m}^2)(9 \text{ m})(0.2 \text{ m}) = 6.48 \text{ kN} \qquad \textit{Ans.}$$

Fig. 4–50

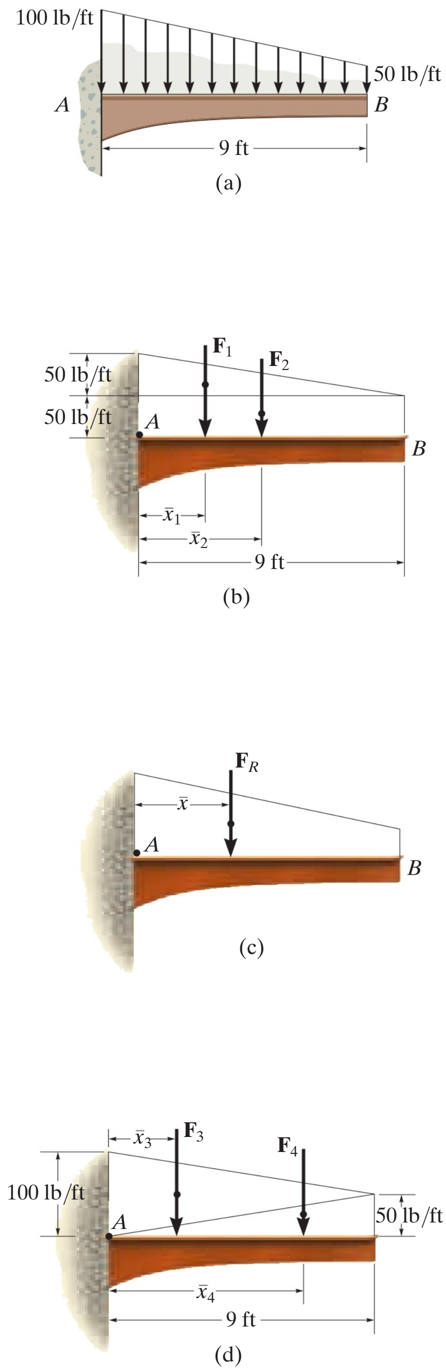


Fig. 4–51

The granular material exerts the distributed loading on the beam as shown in Fig. 4–51*a*. Determine the magnitude and location of the equivalent resultant of this load.

SOLUTION

The area of the loading diagram is a *trapezoid*, and therefore the solution can be obtained directly from the area and centroid formulas for a trapezoid listed on the inside back cover. Since these formulas are not easily remembered, instead we will solve this problem by using "composite" areas. Here we will divide the trapezoidal loading into a rectangular and triangular loading as shown in Fig. 4–51*b*. The magnitude of the force represented by each of these loadings is equal to its associated *area*,

$$F_1 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

$$F_2 = (9\text{ ft})(50\text{ lb/ft}) = 450\text{ lb}$$

The lines of action of these parallel forces act through the respective *centroids* of their associated areas and therefore intersect the beam at

$$\bar{x}_1 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$

$$\bar{x}_2 = \tfrac{1}{2}(9\text{ ft}) = 4.5\text{ ft}$$

The two parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ can be reduced to a single resultant $\mathbf{F}_R$. The magnitude of $\mathbf{F}_R$ is

$$+\downarrow F_R = \Sigma F; \qquad F_R = 225 + 450 = 675\text{ lb} \qquad \textit{Ans.}$$

We can find the location of $\mathbf{F}_R$ with reference to point A, Figs. 4–51*b* and 4–51*c*. We require

$$\circlearrowleft + (M_R)_A = \Sigma M_A; \quad \bar{x}(675) = 3(225) + 4.5(450)$$

$$\bar{x} = 4\text{ ft} \qquad \textit{Ans.}$$

NOTE: The trapezoidal area in Fig. 4–51*a* can also be divided into two triangular areas as shown in Fig. 4–51*d*. In this case

$$F_3 = \tfrac{1}{2}(9\text{ ft})(100\text{ lb/ft}) = 450\text{ lb}$$

$$F_4 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

and

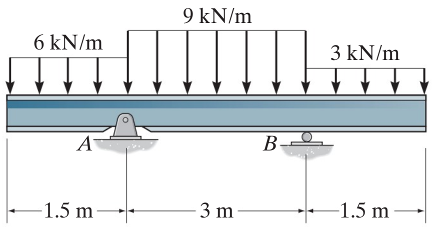
$$\bar{x}_3 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$

$$\bar{x}_4 = 9\text{ ft} - \tfrac{1}{3}(9\text{ ft}) = 6\text{ ft}$$

Using these results, show that again $F_R = 675$ lb and $\bar{x} = 4$ ft.

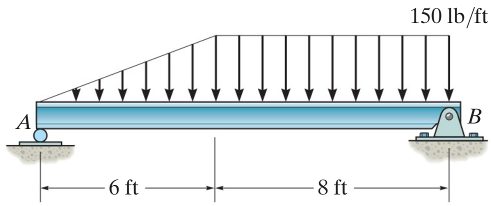
FUNDAMENTAL PROBLEMS

F4–37. Determine the resultant force and specify where it acts on the beam measured from *A*.



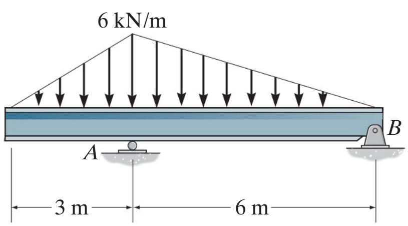
Prob. F4–37

F4–38. Determine the resultant force and specify where it acts on the beam measured from *A*.



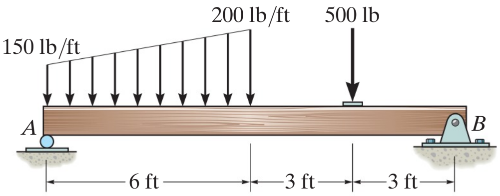
Prob. F4–38

F4–39. Determine the resultant force and specify where it acts on the beam measured from *A*.



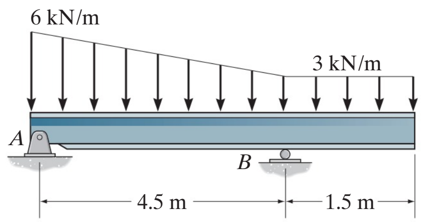
Prob. F4–39

F4–40. Determine the resultant force and specify where it acts on the beam measured from *A*.



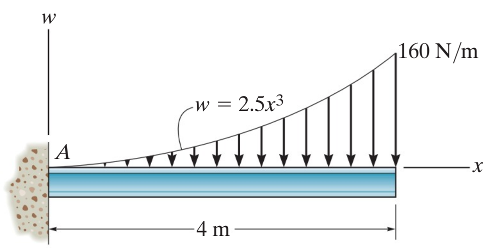
Prob. F4–40

F4–41. Determine the resultant force and specify where it acts on the beam measured from *A*.



Prob. F4–41

F4–42. Determine the resultant force and specify where it acts on the beam measured from *A*.



Prob. F4–42

Lecture 7

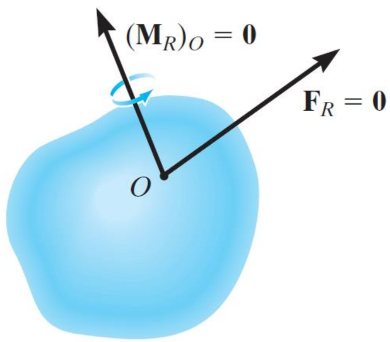
Equilibrium of a Rigid Body

CHAPTER OBJECTIVES

- **To develop the equations of equilibrium for a rigid body.**
- **To introduce the concept of the free-body diagram for a rigid body.**
- **To show how to solve rigid-body equilibrium problems using the equations of equilibrium.**

5.1 Conditions for Rigid-Body Equilibrium

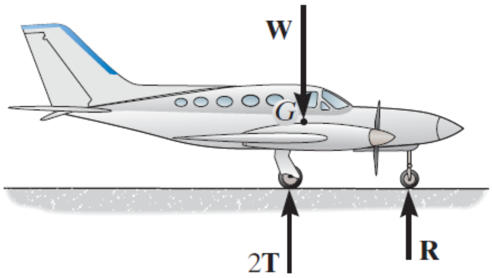
Using the methods of the previous chapter, the force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point *O* on or off the body, Fig.. If this resultant force and couple moment are both equal to zero, then the body is said to be in ***equilibrium***. Mathematically, the equilibrium of a body is expressed as



$$F_R = \sum F = 0$$
$$(M_R)_o = \sum M_o = 0 \qquad \textbf{(5.1)}$$

EQUILIBRIUM IN TWO DIMENSIONS

In the first part of the chapter, we will consider the case where the force system acting on a rigid body lies in or may be projected onto a *single* plane and, furthermore, any couple moments acting on the body are directed perpendicular to this plane. This type of force and couple system is often referred to as a two-dimensional or *coplanar* force system. For example, the airplane in Fig.



5.2 Free-Body Diagrams

Successful application of the equations of equilibrium requires a complete specification of *all* the known and unknown external forces that act *on* the body. The best way to account for these forces is to draw a ***free-body diagram***. This diagram is a sketch of the outlined shape of the body, which represents it as being *isolated* or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show *all* the forces and couple moments that the surroundings exert *on the body* so that these effects can be accounted for when the equations of equilibrium are applied.

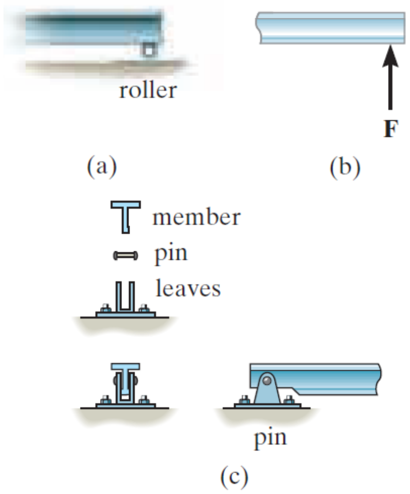
A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.

Support Reactions

Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider the various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems. As a general rule,

- A support prevents the translation of a body in a given direction by exerting a force on the body in the opposite direction.
- A support prevents the rotation of a body in a given direction by exerting a couple moment on the body in the opposite direction.

Table 5–1 lists other common types of supports for bodies subjected to coplanar force systems. (In all cases the angle u is assumed to be known.) Carefully study each of the symbols used to represent these supports and the types of reactions they exert on their contacting members.



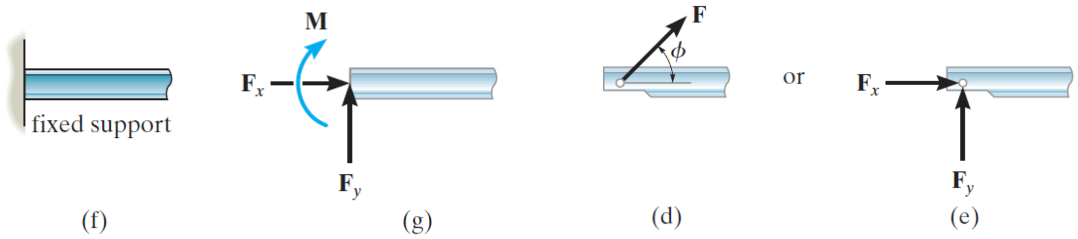
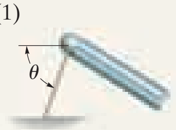
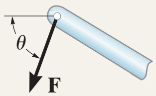
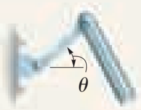
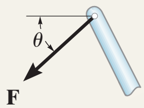
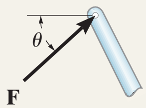
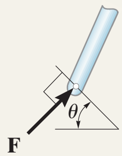
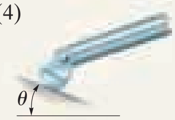
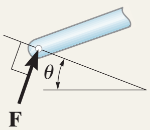
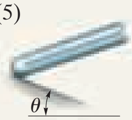
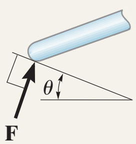
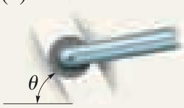
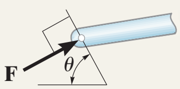
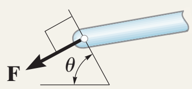
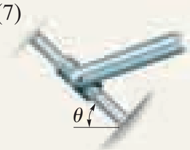
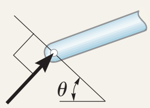
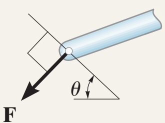
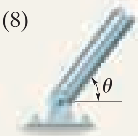
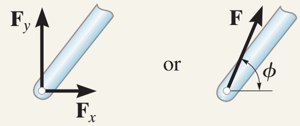
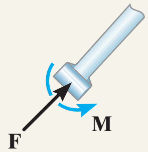
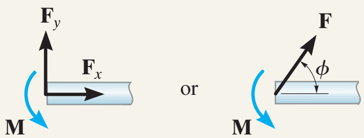


TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

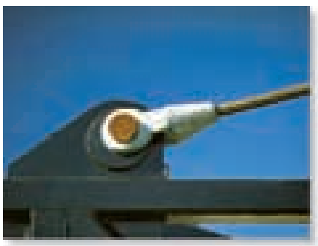
Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

continued

TABLE 5–1 Continued

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Typical examples of actual supports are shown in the following sequence of photos. The numbers refer to the connection types in Table 5–1.



(© Russell C. Hibbeler)

The cable exerts a force on the bracket in the direction of the cable. (1)

The rocker support for this bridge girder allows horizontal movement so the bridge is free to expand and contract due to a change in temperature. (4) (© Russell C. Hibbeler)

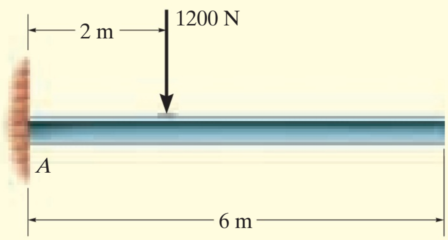
This concrete girder rests on the ledge that is assumed to act as a smooth contacting surface. (5) (© Russell C. Hibbeler)

Typical pin support for a beam. (8) (© Russell C. Hibbeler)

The floor beams of this building are welded together and thus form fixed connections. (10) (© Russell C. Hibbeler)

EXAMPLE 5.1

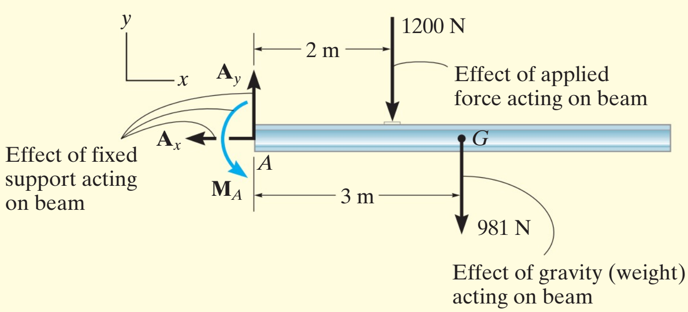
Draw the free-body diagram of the uniform beam shown in Fig. 5–7*a*. The beam has a mass of 100 kg.



(a)

SOLUTION

The free-body diagram of the beam is shown in Fig. 5–7*b*. Since the support at *A* is fixed, the wall exerts three reactions *on the beam*, denoted as $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{M}_A$. The magnitudes of these reactions are *unknown*, and their sense has been *assumed*. The weight of the beam, $W = 100(9.81)\text{ N} = 981\text{ N}$, acts through the beam's center of gravity *G*, which is 3 m from *A* since the beam is uniform.



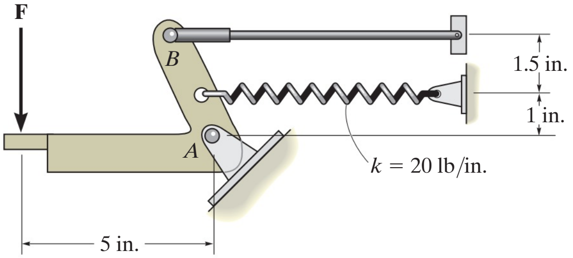
(b)

Fig. 5–7

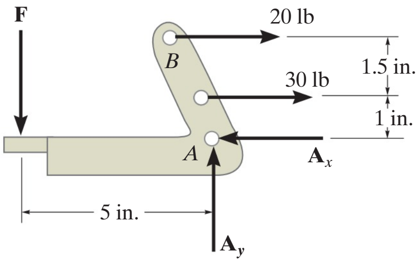
EXAMPLE 5.2

Draw the free-body diagram of the foot lever shown in Fig. 5–8*a*. The operator applies a vertical force to the pedal so that the spring is stretched 1.5 in. and the force on the link at *B* is 20 lb.

(a)



(b)



(c)

Fig. 5–8 (© Russell C. Hibbeler)

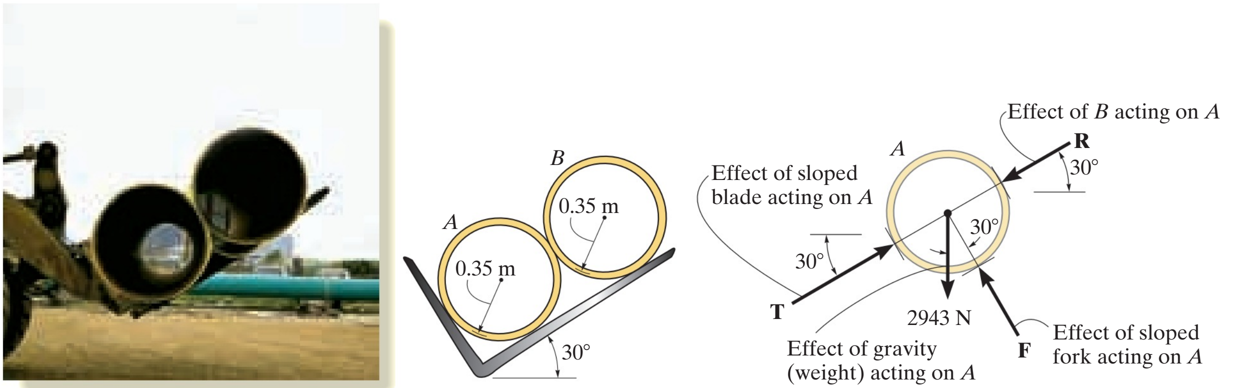
5

SOLUTION

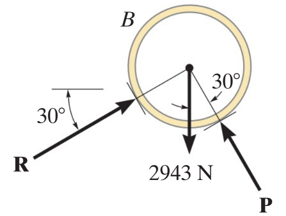
By inspection of the photo the lever is loosely bolted to the frame at *A* and so this bolt acts as a pin. (See (8) in Table 5–1.) Although not shown here the link at *B* is pinned at both ends and so it is like (2) in Table 5–1. After making the proper measurements, the idealized model of the lever is shown in Fig. 5–8*b*. From this, the free-body diagram is shown in Fig. 5–8*c*. Since the pin at *A* is *removed*, it exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ on the lever. The link exerts a force of 20 lb, acting in the direction of the link. In addition the spring also exerts a horizontal force on the lever. If the stiffness is measured and found to be $k = 20$ lb/in., then since the stretch $s = 1.5$ in., using Eq. 3–2, $F_s = ks = 20$ lb/in. (1.5 in.) $= 30$ lb. Finally, the operator's shoe applies a vertical force of $\mathbf{F}$ on the pedal. The dimensions of the lever are also shown on the free-body diagram, since this information will be useful when calculating the moments of the forces. As usual, the senses of the unknown forces at *A* have been assumed. The correct senses will become apparent after solving the equilibrium equations.

EXAMPLE 5.3

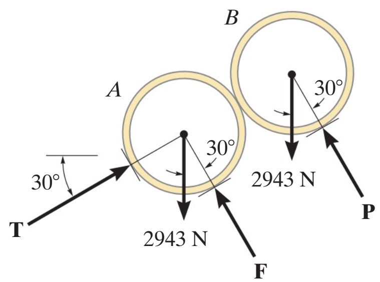
Two smooth pipes, each having a mass of 300 kg, are supported by the forked tines of the tractor in Fig. 5–9*a*. Draw the free-body diagrams for each pipe and both pipes together.



(a)
(© Russell C. Hibbeler)
(b)
(c)



(d)



(e)

Fig. 5–9

SOLUTION

The idealized model from which we must draw the free-body diagrams is shown in Fig. 5–9*b*. Here the pipes are identified, the dimensions have been added, and the physical situation reduced to its simplest form.

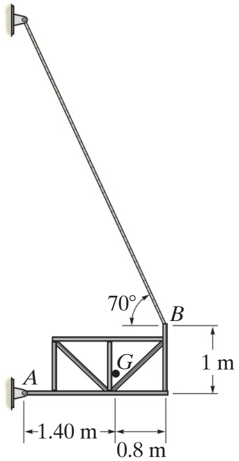
Removing the surfaces of contact, the free-body diagram for pipe *A* is shown in Fig. 5–9*c*. Its weight is $W = 300(9.81)\text{ N} = 2943\text{ N}$. Assuming all contacting surfaces are *smooth*, the reactive forces **T**, **F**, **R** act in a direction *normal* to the tangent at their surfaces of contact.

The free-body diagram of the isolated pipe *B* is shown in Fig. 5–9*d*. Can you identify each of the three forces acting *on this pipe*? In particular, note that **R**, representing the force of *A* on *B*, Fig. 5–9*d*, is equal and opposite to **R** representing the force of *B* on *A*, Fig. 5–9*c*. This is a consequence of Newton's third law of motion.

The free-body diagram of both pipes combined ("system") is shown in Fig. 5–9*e*. Here the contact force **R**, which acts between *A* and *B*, is considered as an *internal* force and hence is not shown on the free-body diagram. That is, it represents a pair of equal but opposite collinear forces which cancel each other.

EXAMPLE 5.4

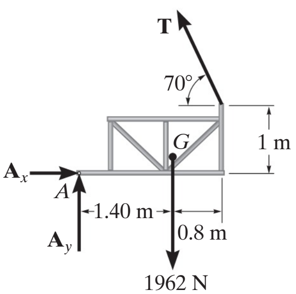
Draw the free-body diagram of the unloaded platform that is suspended off the edge of the oil rig shown in Fig. 5–10*a*. The platform has a mass of 200 kg.



(b)


(a)

Fig. 5–10 (© Russell C. Hibbeler)



(c)

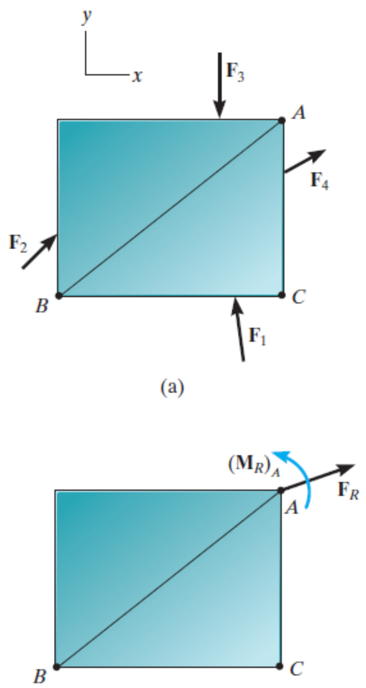
SOLUTION

The idealized model of the platform will be considered in two dimensions because by observation the loading and the dimensions are all symmetrical about a vertical plane passing through its center, Fig. 5–10*b*. The connection at *A* is considered to be a pin, and the cable supports the platform at *B*. The direction of the cable and average dimensions of the platform are listed, and the center of gravity *G* has been determined. It is from this model that we have drawn the free-body diagram shown in Fig. 5–10*c*. The platform's weight is $200(9.81) = 1962$ N. The supports have been *removed*, and the force components $\mathbf{A}_x$ and $\mathbf{A}_y$ along with the cable force $\mathbf{T}$ represent the reactions that *both* pins and *both* cables exert on the platform, Fig. 5–10*a*. As a result, half their magnitudes are developed on each side of the platform.

5.3 Equations of Equilibrium

In Sec. 5.1 we developed the two equations which are both necessary and sufficient for the equilibrium of a rigid body, namely, $\sum \mathbf{F} = \mathbf{0}$ and $\sum \mathbf{M}_O = \mathbf{0}$. When the body is subjected to a system of forces, which all lie in the x–y plane, then the forces can be resolved into their x and y components. Consequently, the conditions for equilibrium in two dimensions are

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0 \end{aligned} \qquad \textbf{(5.2)}$$



Alternative Sets of Equilibrium Equations. Although Eqs 5.2 are most often used for solving coplanar equilibrium problems, two alternative sets of three independent equilibrium equations may also be used. One such set is

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \end{aligned} \qquad \textbf{(5.3)}$$

When using these equations it is required that a line passing through points A and B is not parallel to the y axis. To prove that Eqs. 5.3 provide the conditions for equilibrium, consider the free-body diagram of the plate.

EXAMPLE 5.5

Determine the horizontal and vertical components of reaction on the beam caused by the pin at *B* and the rocker at *A* as shown in Fig. 5–12*a*. Neglect the weight of the beam.

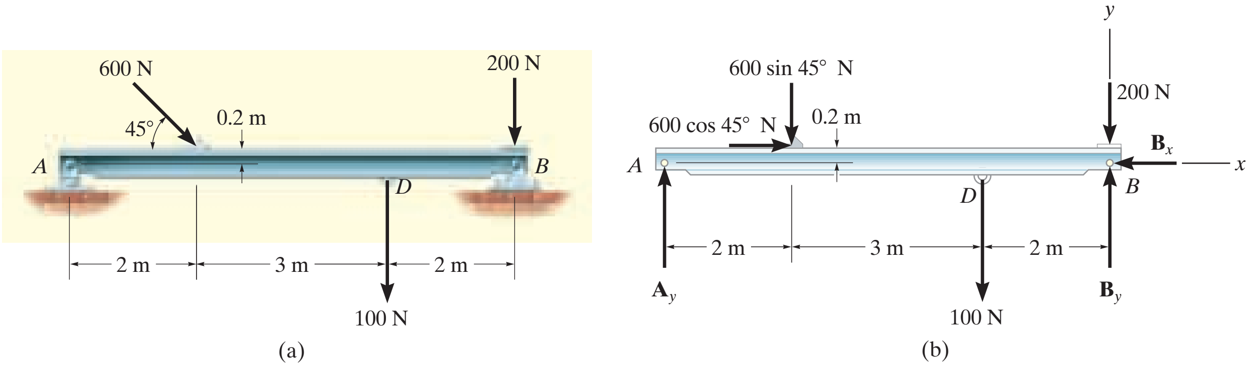
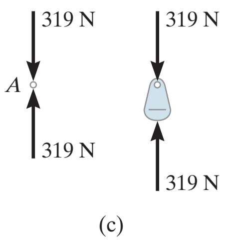


Fig. 5–12



SOLUTION

Free-Body Diagram. The supports are *removed*, and the free-body diagram of the beam is shown in Fig. 5–12*b*. (See Example 5.1.) For simplicity, the 600-N force is represented by its *x* and *y* components as shown in Fig. 5–12*b*.

Equations of Equilibrium. Summing forces in the *x* direction yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 600 \cos 45^\circ \text{ N} - B_x = 0$$

$$B_x = 424 \text{ N} \qquad \textit{Ans.}$$

A direct solution for $\mathbf{A}_y$ can be obtained by applying the moment equation $\Sigma M_B = 0$ about point *B*.

$$\circlearrowleft + \Sigma M_B = 0; \qquad 100 \text{ N}(2 \text{ m}) + (600 \sin 45^\circ \text{ N})(5 \text{ m})$$

$$- (600 \cos 45^\circ \text{ N})(0.2 \text{ m}) - A_y(7 \text{ m}) = 0$$

$$A_y = 319 \text{ N} \qquad \textit{Ans.}$$

Summing forces in the *y* direction, using this result, gives

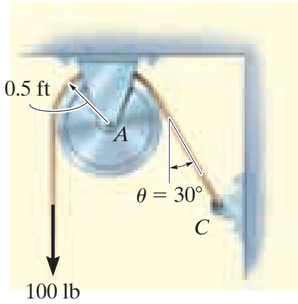
$$+\uparrow \Sigma F_y = 0; \qquad 319 \text{ N} - 600 \sin 45^\circ \text{ N} - 100 \text{ N} - 200 \text{ N} + B_y = 0$$

$$B_y = 405 \text{ N} \qquad \textit{Ans.}$$

NOTE: Remember, the support forces in Fig. 5–12*b* are the result of pins that *act on the beam*. The opposite forces act on the pins. For example, Fig. 5–12*c* shows the equilibrium of the pin at *A* and the rocker.

EXAMPLE 5.6

The cord shown in Fig. 5–13*a* supports a force of 100 lb and wraps over the frictionless pulley. Determine the tension in the cord at *C* and the horizontal and vertical components of reaction at pin *A*.

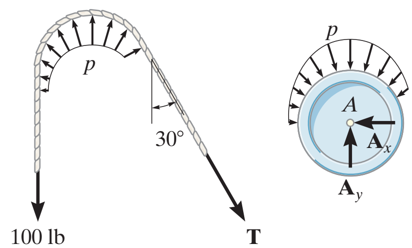


(a)

Fig. 5–13

SOLUTION

Free-Body Diagrams. The free-body diagrams of the cord and pulley are shown in Fig. 5–13*b*. Note that the principle of action, equal but opposite reaction must be carefully observed when drawing each of these diagrams: the cord exerts an unknown load distribution *p* on the pulley at the contact surface, whereas the pulley exerts an equal but opposite effect on the cord. For the solution, however, it is simpler to *combine* the free-body diagrams of the pulley and this portion of the cord, so that the distributed load becomes *internal* to this "system" and is therefore eliminated from the analysis, Fig. 5–13*c*.



(b)

Equations of Equilibrium. Summing moments about point *A* to eliminate $\mathbf{A}_x$ and $\mathbf{A}_y$, Fig. 5–13*c*, we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad 100 \text{ lb } (0.5 \text{ ft}) - T(0.5 \text{ ft}) = 0$$

$$T = 100 \text{ lb} \qquad \textit{Ans.}$$

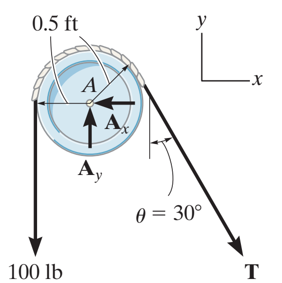
Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad -A_x + 100 \sin 30° \text{ lb} = 0$$

$$A_x = 50.0 \text{ lb} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 100 \text{ lb} - 100 \cos 30° \text{ lb} = 0$$

$$A_y = 187 \text{ lb} \qquad \textit{Ans.}$$



(c)

NOTE: From the moment equation, it is seen that the tension remains *constant* as the cord passes over the pulley. (This of course is true for *any angle* θ at which the cord is directed and for *any radius r* of the pulley.)

EXAMPLE 5.7

The member shown in Fig. 5–14*a* is pin connected at *A* and rests against a smooth support at *B*. Determine the horizontal and vertical components of reaction at the pin *A*.

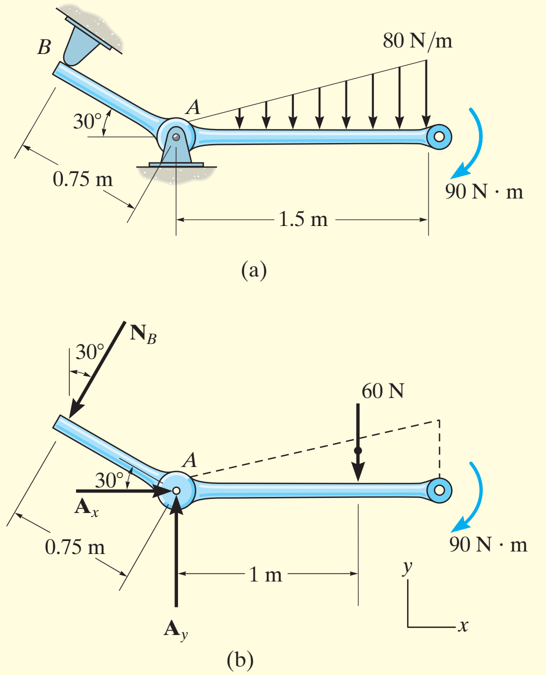


Fig. 5–14

SOLUTION

Free-Body Diagram. As shown in Fig. 5–14*b*, the supports are removed and the reaction $\mathbf{N}_B$ is perpendicular to the member at *B*. Also, horizontal and vertical components of reaction are represented at *A*. The resultant of the distributed loading is $\frac{1}{2}(1.5\text{ m})(80\text{ N/m}) = 60\text{ N}$. It acts through the centroid of the triangle, 1 m from *A* as shown.

Equations of Equilibrium. Summing moments about *A*, we obtain a direct solution for N_B,

$$\circlearrowleft +\Sigma M_A = 0; \quad -90\text{ N}\cdot\text{m} - 60\text{ N}(1\text{ m}) + N_B(0.75\text{ m}) = 0$$

$$N_B = 200\text{ N}$$

Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x - 200\sin 30^\circ\text{ N} = 0$$

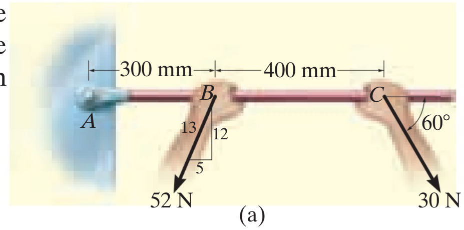
$$A_x = 100\text{ N} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 200\cos 30^\circ\text{ N} - 60\text{ N} = 0$$

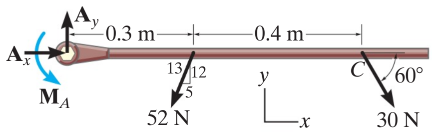
$$A_y = 233\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 5.8

The box wrench in Fig. 5–15*a* is used to tighten the bolt at *A*. If the wrench does not turn when the load is applied to the handle, determine the torque or moment applied to the bolt and the force of the wrench on the bolt.



(a)



(b)

Fig. 5–15

SOLUTION

Free-Body Diagram. The free-body diagram for the wrench is shown in Fig. 5–15*b*. Since the bolt acts as a "fixed support," when it is removed, it exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ and a moment $\mathbf{M}_A$ on the wrench at *A*.

Equations of Equilibrium.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad A_x - 52\left(\tfrac{5}{13}\right)\text{N} + 30\cos 60^\circ\,\text{N} = 0$$

$$A_x = 5.00\ \text{N} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 52\left(\tfrac{12}{13}\right)\text{N} - 30\sin 60^\circ\,\text{N} = 0$$

$$A_y = 74.0\ \text{N} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_A = 0; \quad M_A - \left[52\left(\tfrac{12}{13}\right)\text{N}\right](0.3\ \text{m}) - (30\sin 60^\circ\ \text{N})(0.7\ \text{m}) = 0$$

$$M_A = 32.6\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Note that $\mathbf{M}_A$ must be *included* in this moment summation. This couple moment is a free vector and represents the twisting resistance of the bolt on the wrench. By Newton's third law, the wrench exerts an equal but opposite moment or torque on the bolt. Furthermore, the resultant force on the wrench is

$$F_A = \sqrt{(5.00)^2 + (74.0)^2} = 74.1\ \text{N} \qquad \textit{Ans.}$$

NOTE: Although only *three* independent equilibrium equations can be written for a rigid body, it is a good practice to *check* the calculations using a fourth equilibrium equation. For example, the above computations may be verified in part by summing moments about point *C*:

$$\circlearrowleft + \Sigma M_C = 0; \quad \left[52\left(\tfrac{12}{13}\right)\text{N}\right](0.4\ \text{m}) + 32.6\ \text{N}\cdot\text{m} - 74.0\ \text{N}(0.7\ \text{m}) = 0$$

$$19.2\ \text{N}\cdot\text{m} + 32.6\ \text{N}\cdot\text{m} - 51.8\ \text{N}\cdot\text{m} = 0$$

EXAMPLE 5.9

Determine the horizontal and vertical components of reaction on the member at the pin A, and the normal reaction at the roller B in Fig. 5–16*a*.

SOLUTION

Free-Body Diagram. All the supports are removed and so the free-body diagram is shown in Fig. 5–16*b*. The pin at A exerts two components of reaction on the member, $\mathbf{A}_x$ and $\mathbf{A}_y$.

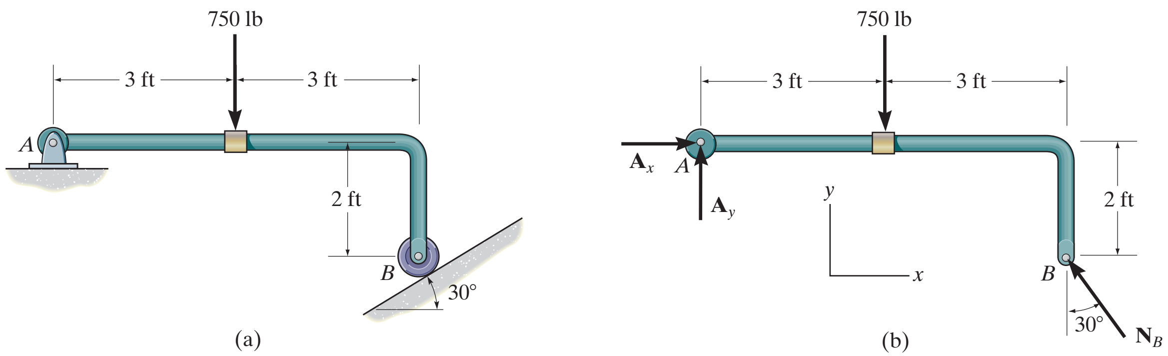
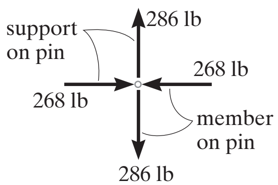


Fig. 5–16



(c)

Equations of Equilibrium. The reaction N_B can be obtained *directly* by summing moments about point A, since $\mathbf{A}_x$ and $\mathbf{A}_y$ produce no moment about A.

$$\circlearrowleft + \Sigma M_A = 0;$$

$$[N_B \cos 30°](6 \text{ ft}) - [N_B \sin 30°](2 \text{ ft}) - 750 \text{ lb}(3 \text{ ft}) = 0$$

$$N_B = 536.2 \text{ lb} = 536 \text{ lb} \qquad \textit{Ans.}$$

Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x - (536.2 \text{ lb}) \sin 30° = 0$$

$$A_x = 268 \text{ lb} \qquad \textit{Ans.}$$

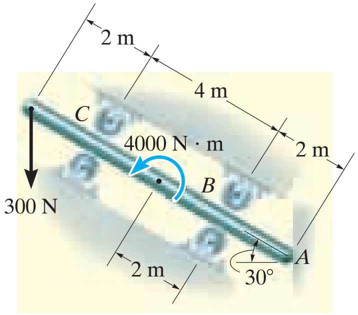
$$+\uparrow \Sigma F_y = 0; \qquad A_y + (536.2 \text{ lb}) \cos 30° - 750 \text{ lb} = 0$$

$$A_y = 286 \text{ lb} \qquad \textit{Ans.}$$

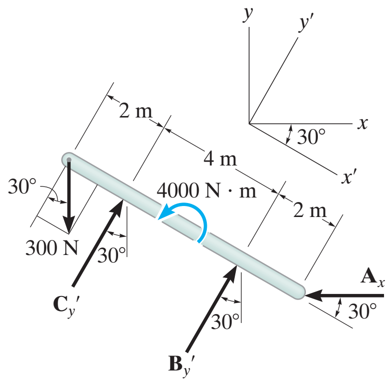
Details of the equilibrium of the pin at A are shown in Fig. 5–16*c*.

EXAMPLE 5.10

The uniform smooth rod shown in Fig. 5–17*a* is subjected to a force and couple moment. If the rod is supported at *A* by a smooth wall and at *B* and *C* either at the top or bottom by rollers, determine the reactions at these supports. Neglect the weight of the rod.



(a)



(b)

Fig. 5–17

SOLUTION

Free-Body Diagram. Removing the supports as shown in Fig. 5–17*b*, all the reactions act normal to the surfaces of contact since these surfaces are smooth. The reactions at *B* and *C* are shown acting in the positive y' direction. This assumes that only the rollers located on the bottom of the rod are used for support.

Equations of Equilibrium. Using the *x*, *y* coordinate system in Fig. 5–17*b*, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad C_{y'} \sin 30^\circ + B_{y'} \sin 30^\circ - A_x = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0; \qquad -300\text{ N} + C_{y'} \cos 30^\circ + B_{y'} \cos 30^\circ = 0 \qquad (2)$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad -B_{y'}(2\text{ m}) + 4000\text{ N}\cdot\text{m} - C_{y'}(6\text{ m}) + (300 \cos 30^\circ\text{ N})(8\text{ m}) = 0 \qquad (3)$$

When writing the moment equation, it should be noted that the line of action of the force component 300 sin 30° N passes through point *A*, and therefore this force is not included in the moment equation.

Solving Eqs. 2 and 3 simultaneously, we obtain

$$B_{y'} = -1000.0\text{ N} = -1\text{ kN} \qquad \textit{Ans.}$$

$$C_{y'} = 1346.4\text{ N} = 1.35\text{ kN} \qquad \textit{Ans.}$$

Since $B_{y'}$ is a negative scalar, the sense of $\mathbf{B}_{y'}$ is opposite to that shown on the free-body diagram in Fig. 5–17*b*. Therefore, the top roller at *B* serves as the support rather than the bottom one. Retaining the negative sign for $B_{y'}$ (Why?) and substituting the results into Eq. 1, we obtain

$$1346.4 \sin 30^\circ\text{ N} + (-1000.0 \sin 30^\circ\text{ N}) - A_x = 0$$

$$A_x = 173\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 5.11

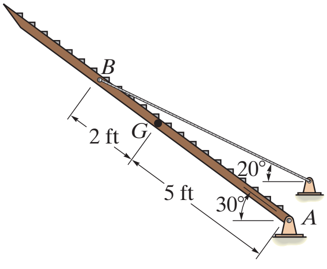
(a)

(© Russell C. Hibbeler)

The uniform truck ramp shown in Fig. 5–18*a* has a weight of 400 lb and is pinned to the body of the truck at each side and held in the position shown by the two side cables. Determine the tension in the cables.

SOLUTION

The idealized model of the ramp, which indicates all necessary dimensions and supports, is shown in Fig. 5–18*b*. Here the center of gravity is located at the midpoint since the ramp is considered to be uniform.



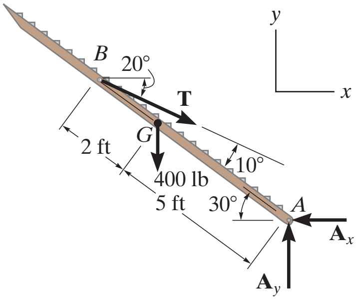
(b)

Free-Body Diagram. Removing the supports from the idealized model, the ramp's free-body diagram is shown in Fig. 5–18*c*.

Equations of Equilibrium. Summing moments about point A will yield a direct solution for the cable tension. Using the principle of moments, there are several ways of determining the moment of $\mathbf{T}$ about A. If we use x and y components, with $\mathbf{T}$ applied at B, we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad -T\cos 20°(7\sin 30°\text{ ft}) + T\sin 20°(7\cos 30°\text{ ft})$$
$$+ 400\text{ lb }(5\cos 30°\text{ ft}) = 0$$
$$T = 1425\text{ lb}$$

We can also determine the moment of $\mathbf{T}$ about A by resolving it into components along and perpendicular to the ramp at B. Then the moment of the component along the ramp will be zero about A, so that



(c)

Fig. 5–18

$$\circlearrowleft + \Sigma M_A = 0; \qquad -T\sin 10°(7\text{ ft}) + 400\text{ lb }(5\cos 30°\text{ ft}) = 0$$
$$T = 1425\text{ lb}$$

Since there are two cables supporting the ramp,

$$T' = \frac{T}{2} = 712\text{ lb} \qquad \textit{Ans.}$$

NOTE: As an exercise, show that $A_x = 1339$ lb and $A_y = 887$ lb.

EXAMPLE 5.12

Determine the support reactions on the member in Fig. 5–19*a*. The collar at A is fixed to the member and can slide vertically along the vertical shaft.

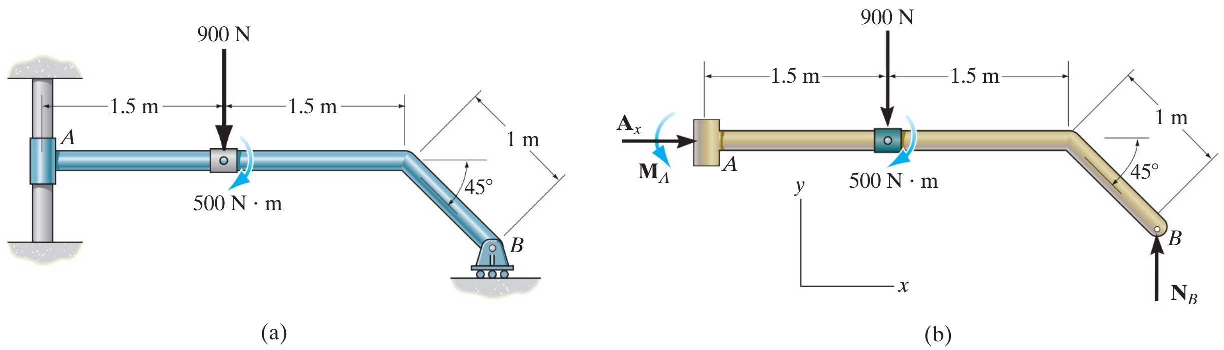


Fig. 5–19

SOLUTION

Free-Body Diagram. Removing the supports, the free-body diagram of the member is shown in Fig. 5–19*b*. The collar exerts a horizontal force $\mathbf{A}_x$ and moment $\mathbf{M}_A$ on the member. The reaction $\mathbf{N}_B$ of the roller on the member is vertical.

Equations of Equilibrium. The forces A_x and N_B can be determined directly from the force equations of equilibrium.

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x = 0 \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad N_B - 900\text{ N} = 0$$

$$N_B = 900\text{ N} \qquad \textit{Ans.}$$

The moment M_A can be determined by summing moments either about point A or point B.

$$\circlearrowleft + \Sigma M_A = 0;$$

$$M_A - 900\text{ N}(1.5\text{ m}) - 500\text{ N}\cdot\text{m} + 900\text{ N}\,[3\text{ m} + (1\text{ m})\cos 45°] = 0$$

$$M_A = -1486\text{ N}\cdot\text{m} = 1.49\text{ kN}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}$$

or

$$\circlearrowleft + \Sigma M_B = 0; \quad M_A + 900\text{ N}\,[1.5\text{ m} + (1\text{ m})\cos 45°] - 500\text{ N}\cdot\text{m} = 0$$

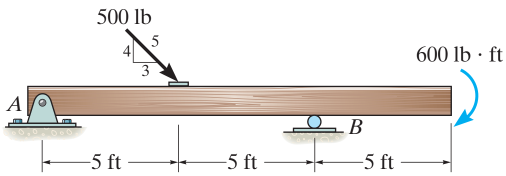
$$M_A = -1486\text{ N}\cdot\text{m} = 1.49\text{ kN}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{M}_A$ has the opposite sense of rotation to that shown on the free-body diagram.

FUNDAMENTAL PROBLEMS

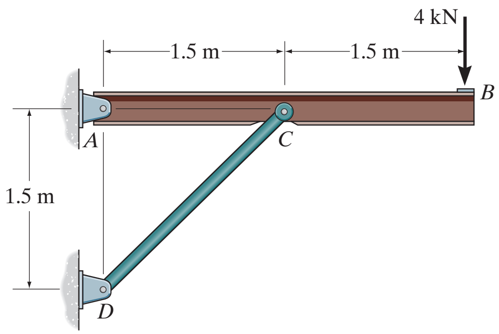
All problem solutions must include an FBD.

F5–1. Determine the horizontal and vertical components of reaction at the supports. Neglect the thickness of the beam.



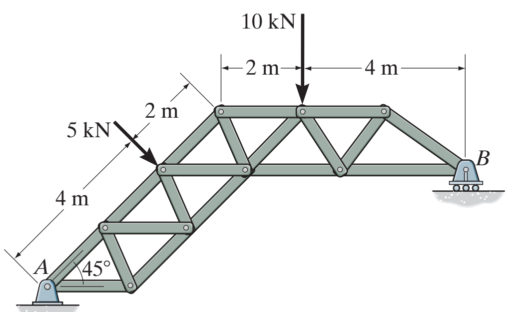
Prob. F5–1

F5–2. Determine the horizontal and vertical components of reaction at the pin A and the reaction on the beam at C.



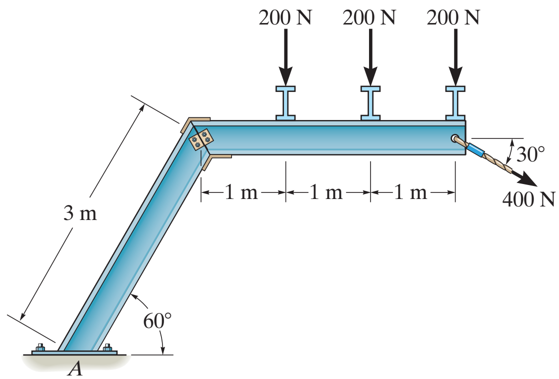
Prob. F5–2

F5–3. The truss is supported by a pin at A and a roller at B. Determine the support reactions.



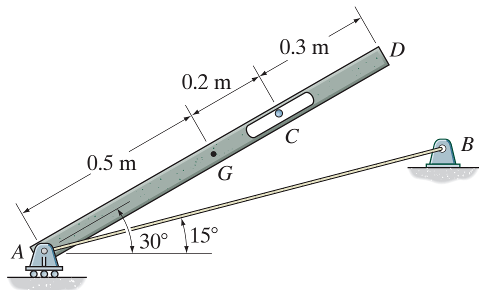
Prob. F5–3

F5–4. Determine the components of reaction at the fixed support A. Neglect the thickness of the beam.



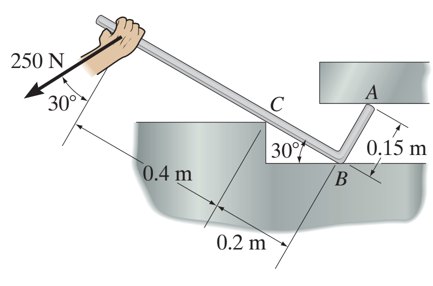
Prob. F5–4

F5–5. The 25-kg bar has a center of mass at G. If it is supported by a smooth peg at C, a roller at A, and cord AB, determine the reactions at these supports.



Prob. F5–5

F5–6. Determine the reactions at the smooth contact points A, B, and C on the bar.



Prob. F5–6

Lecture 8

Center of Gravity and Centroid

CHAPTER OBJECTIVES

- To discuss the concept of the center of gravity, center of mass, and the centroid.
- To show how to determine the location of the center of gravity and centroid for a body of arbitrary shape and one composed of composite parts.
- To use the theorems of Pappus and Guldinus for finding the surface area and volume for a body having axial symmetry.
- To present a method for finding the resultant of a general distributed loading and to show how it applies to finding the resultant force of a pressure loading caused by a fluid.

9.1 Center of Gravity, Center of Mass, and the Centroid of a Body

Knowing the resultant or total weight of a body and its location is important when considering the effect this force produces on the body. The point of location is called the center of gravity, and in this section we will show how to find it for an irregularly shaped body. We will then extend this method to show how to find the body's center of mass, and its geometric center or centroid.

Center of Gravity. A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW. These weights will form a parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the ***center of gravity***, G*.

*In a strict sense this is true as long as the gravity field is assumed to have the same magnitude and direction everywhere. Although the actual force of gravity is directed toward the center of the earth, and this force varies with its distance from the center, for most engineering applications we can assume the gravity field has the same magnitude and direction everywhere.

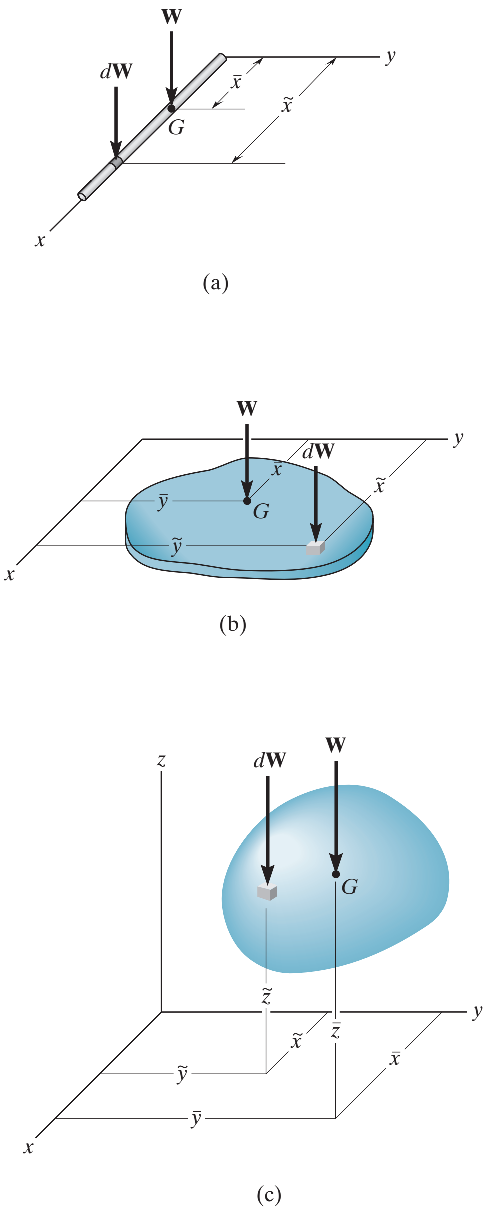


Fig. 9–1

To show how to determine the location of the center of gravity, consider the rod in Fig. 9–1*a*, where the segment having the weight dW is located at the arbitrary position $\tilde{x}$. Using the methods outlined in Sec. 4.8, the total weight of the rod is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \qquad W = \int dW$$

The location of the center of gravity, measured from the y axis, is determined by equating the moment of W about the y axis, Fig. 9–1*b*, to the sum of the moments of the weights of all its particles about this same axis. Therefore,

$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}\,dW$$

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW}$$

In a similar manner, if the body represents a plate, Fig. 9–1*b*, then a moment balance about the x and y axes would be required to determine the location $(\bar{x}, \bar{y})$ of point G. Finally we can generalize this idea to a three-dimensional body, Fig. 9–1*c*, and perform a moment balance about all three axes to locate G for any rotated position of the axes. This results in the following equations.

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW} \qquad (9\text{–}1)$$

where

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G.
$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of an arbitrary particle in the body.

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's ***center of mass*** C_m, Fig. 9–2. This location can be determined by substituting $dW = g\, dm$ into Eqs. 9–1. Provided g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\, dm}{\int dm} \tag{9–2}$$

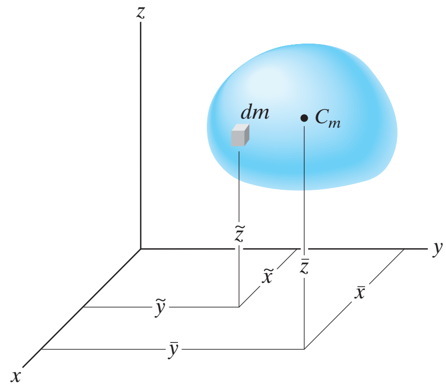


Fig. 9–2

Centroid of a Volume. If the body in Fig. 9–3 is made from a *homogeneous material*, then its density ρ (rho) will be *constant*. Therefore, a differential element of volume dV has a mass $dm = \rho\, dV$. Substituting this into Eqs. 9–2 and canceling out ρ, we obtain formulas that locate the ***centroid*** C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\, dV}{\int_V dV} \tag{9–3}$$

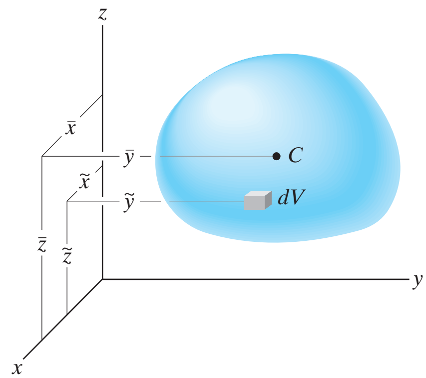


Fig. 9–3

These equations represent a balance of the moments of the volume of the body. Therefore, if the volume possesses two planes of symmetry, then its centroid must lie along the line of intersection of these two planes. For example, the cone in Fig. 9–4 has a centroid that lies on the y axis so that $\bar{x} = \bar{z} = 0$. The location $\bar{y}$ can be found using a single integration by choosing a differential element represented by a *thin disk* having a thickness dy and radius $r = z$. Its volume is $dV = \pi r^2\, dy = \pi z^2\, dy$ and its centroid is at $\tilde{x} = 0$, $\tilde{y} = y$, $\tilde{z} = 0$.

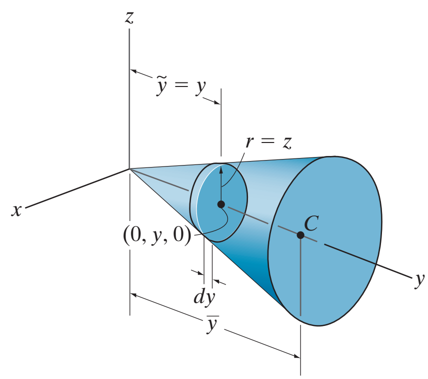


Fig. 9–4

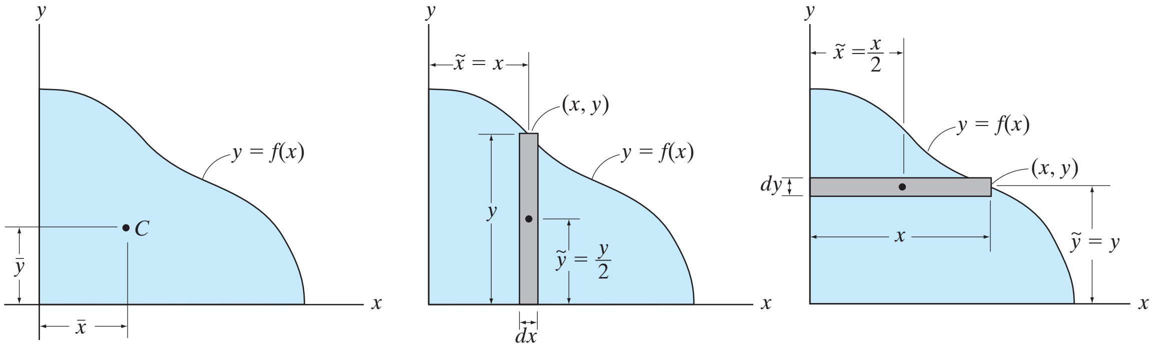


Fig. 9–5

Integration must be used to determine the location of the center of gravity of this lamp post due to the curvature of the member. (© Russell C. Hibbeler)

Centroid of an Area. If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 9–5*a*, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 9–3, namely,

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \tag{9–4}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. 9–5*b*, the area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$. If we consider a horizontal strip, Fig. 9–5*c*, then $dA = x\,dy$, and its centroid is located at $\tilde{x} = x/2$ and $\tilde{y} = y$.

Centroid of a Line. If a line segment (or rod) lies within the x–y plane and it can be described by a thin curve $y = f(x)$, Fig. 9–6*a*, then its centroid is determined from

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} \tag{9–5}$$

Here, the length of the differential element is given by the Pythagorean theorem, $dL = \sqrt{(dx)^2 + (dy)^2}$, which can also be written in the form

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

or

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2}$$

$$= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

Either one of these expressions can be used; however, for application, the one that will result in a simpler integration should be selected. For example, consider the rod in Fig. 9–6*b*, defined by $y = 2x^2$. The length of the element is $dL = \sqrt{1 + (dy/dx)^2}\, dx$, and since $dy/dx = 4x$, then $dL = \sqrt{1 + (4x)^2}\, dx$. The centroid for this element is located at $\tilde{x} = x$ and $\tilde{y} = y$.

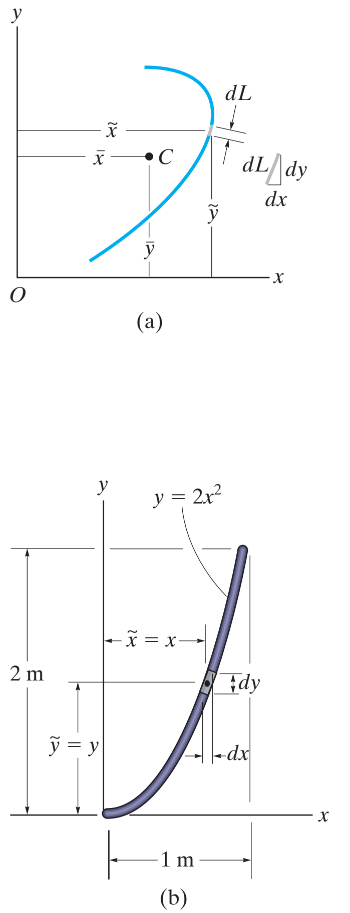


Fig. 9–6

Important Points

- The centroid represents the geometric center of a body. This point coincides with the center of mass or the center of gravity only if the material composing the body is uniform or homogeneous.
- Formulas used to locate the center of gravity or the centroid simply represent a balance between the sum of moments of all the parts of the system and the moment of the "resultant" for the system.
- In some cases the centroid is located at a point that is not on the object, as in the case of a ring, where the centroid is at its center. Also, this point will lie on any axis of symmetry for the body, Fig. 9–7.

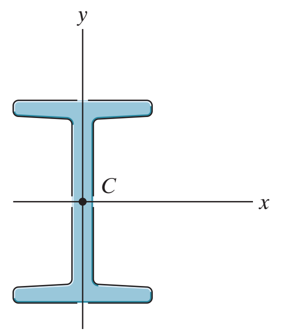


Fig. 9–7

Procedure for Analysis

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For lines the element is represented by a differential line segment of length dL.
- For areas the element is generally a rectangle of area dA, having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

- Express the length dL, area dA, or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

- Substitute the formulations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and dL, dA, or dV into the appropriate equations (Eqs. 9–1 through 9–5).
- Express the function in the integrand in terms of the *same variable as the differential thickness of the element*.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.

EXAMPLE 9.1

Locate the centroid of the rod bent into the shape of a parabolic arc as shown in Fig. 9–8.

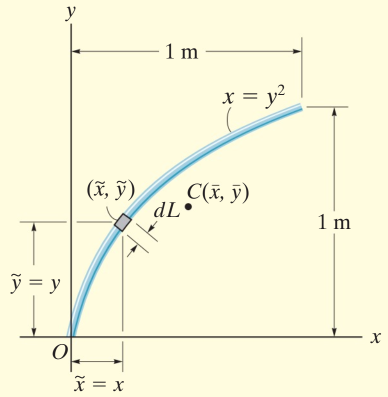


Fig. 9–8

SOLUTION

Differential Element. The differential element is shown in Fig. 9–8. It is located on the curve at the *arbitrary point* (x, y).

Area and Moment Arms. The differential element of length dL can be expressed in terms of the differentials dx and dy using the Pythagorean theorem.

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

Since $x = y^2$, then $dx/dy = 2y$. Therefore, expressing dL in terms of y and dy, we have

$$dL = \sqrt{(2y)^2 + 1}\, dy$$

As shown in Fig. 9–8, the centroid of the element is located at $\tilde{x} = x$, $\tilde{y} = y$.

Integrations. Applying Eq. 9–5 and using the integration formula to evaluate the integrals, we get

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} x\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{\int_0^{1\text{ m}} y^2\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy}$$

$$= \frac{0.6063}{1.479} = 0.410\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} y\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{0.8484}{1.479} = 0.574\text{ m} \qquad \textit{Ans.}$$

NOTE: These results for C seem reasonable when they are plotted on Fig. 9–8.

9

EXAMPLE 9.2

Locate the centroid of the circular wire segment shown in Fig. 9–9.

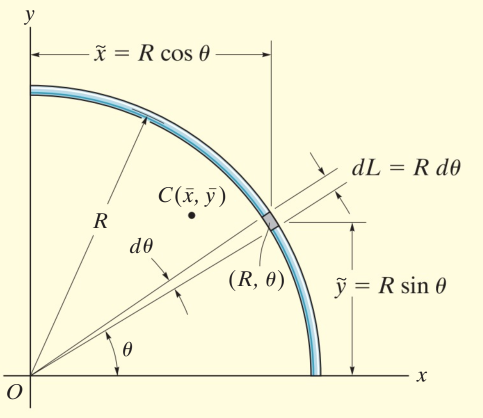


Fig. 9–9

SOLUTION

Polar coordinates will be used to solve this problem since the arc is circular.

Differential Element. A differential circular arc is selected as shown in the figure. This element lies on the curve at (R, θ).

Length and Moment Arm. The length of the differential element is $dL = R\,d\theta$, and its centroid is located at $\tilde{x} = R\cos\theta$ and $\tilde{y} = R\sin\theta$.

Integrations. Applying Eqs. 9–5 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\cos\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2}\cos\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\sin\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2}\sin\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

NOTE: As expected, the two coordinates are numerically the same due to the symmetry of the wire.

9

EXAMPLE 9.3

Determine the distance $\bar{y}$ measured from the x axis to the centroid of the area of the triangle shown in Fig. 9–10.

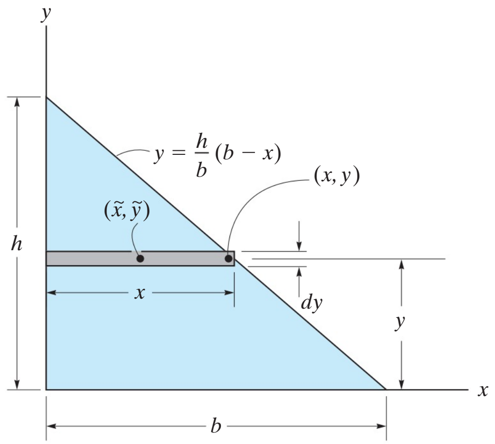


Fig. 9–10

SOLUTION

Differential Element. Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), Fig. 9–10.

Area and Moment Arms. The area of the element is $dA = x\,dy = \frac{b}{h}(h - y)\,dy$, and its centroid is located a distance $\tilde{y} = y$ from the x axis.

Integration. Applying the second of Eqs. 9–4 and integrating with respect to y yields

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^h y\left[\frac{b}{h}(h - y)\,dy\right]}{\int_0^h \frac{b}{h}(h - y)\,dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}$$

$$= \frac{h}{3}$$ *Ans.*

NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

EXAMPLE 9.4

Locate the centroid for the area of a quarter circle shown in Fig. 9–11.

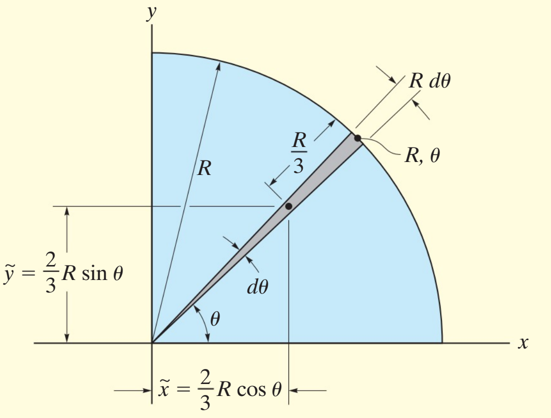


Fig. 9–11

SOLUTION

Differential Element. Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9–11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point (R, θ).

Area and Moment Arms. The area of the element is

$$dA = \tfrac{1}{2}(R)(R\,d\theta) = \frac{R^2}{2}d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at $\tilde{x} = \frac{2}{3}R\cos\theta$, $\tilde{y} = \frac{2}{3}R\sin\theta$.

Integrations. Applying Eqs. 9–4 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\cos\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\cos\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\sin\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\sin\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

EXAMPLE 9.5

Locate the centroid of the area shown in Fig. 9–12*a*.

SOLUTION I

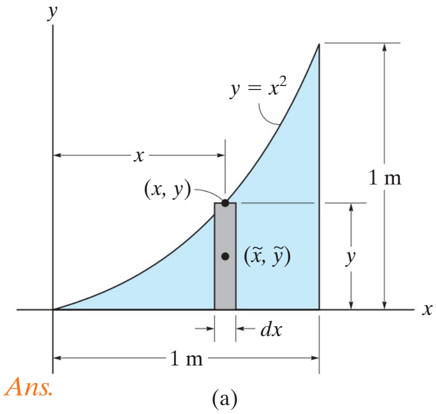
Differential Element. A differential element of thickness dx is shown in Fig. 9–12*a*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 9–4 and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$



SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 9–12*b*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\,dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1-x}{2}\right) = \frac{1+x}{2},\ \tilde{y} = y$$

Integrations. Applying Eqs. 9–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1+x)/2](1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1-y)\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

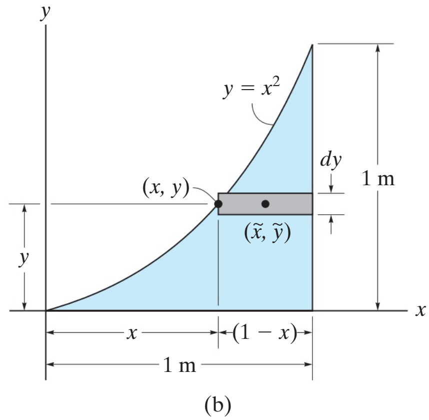


Fig. 9–12

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.

Locate the centroid of the semi-elliptical area shown in Fig. 9–13*a*.

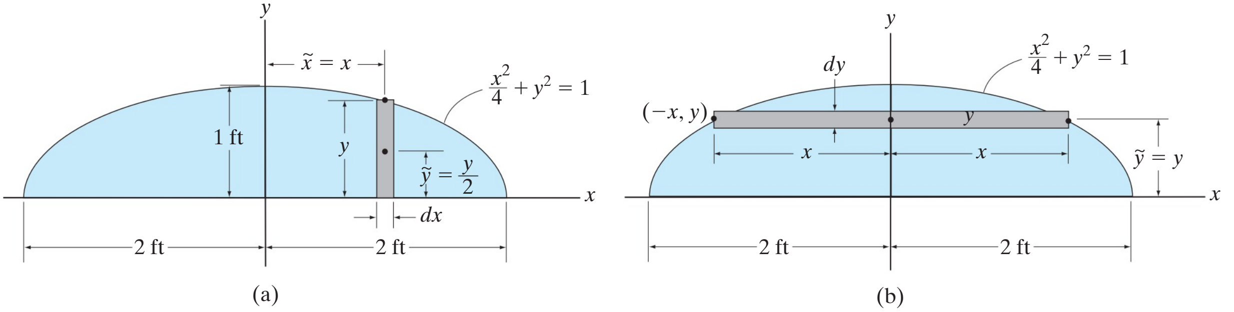


Fig. 9–13

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 9–13*a* will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Applying the second of Eqs. 9–4 with $y = \sqrt{1 - \frac{x^2}{4}}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_{-2\text{ ft}}^{2\text{ ft}} \frac{y}{2}(y\,dx)}{\int_{-2\text{ ft}}^{2\text{ ft}} y\,dx} = \frac{\frac{1}{2}\int_{-2\text{ ft}}^{2\text{ ft}}\left(1 - \frac{x^2}{4}\right)dx}{\int_{-2\text{ ft}}^{2\text{ ft}}\sqrt{1 - \frac{x^2}{4}}\,dx} = \frac{4/3}{\pi} = 0.424\text{ ft} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 9–13*b*.

Area and Moment Arms. The area is $dA = 2x\,dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ ft}} y(2x\,dy)}{\int_0^{1\text{ ft}} 2x\,dy} = \frac{\int_0^{1\text{ ft}} 4y\sqrt{1 - y^2}\,dy}{\int_0^{1\text{ ft}} 4\sqrt{1 - y^2}\,dy} = \frac{4/3}{\pi}\text{ ft} = 0.424\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 9.7

Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 9–14.

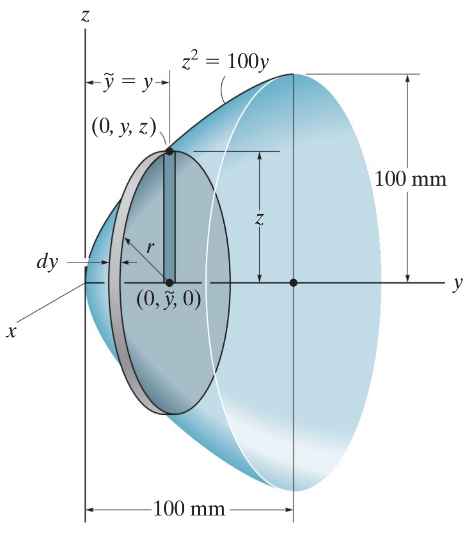


Fig. 9–14

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\,dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–3 and integrating with respect to y yields.

$$\bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\,dy}{\int_0^{100\text{ mm}} (\pi z^2)\,dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\,dy}{100\pi \int_0^{100\text{ mm}} y\,dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

Determine the location of the center of mass of the cylinder shown in Fig. 9–15 if its density varies directly with the distance from its base, i.e., $\rho = 200z \text{ kg/m}^3$.

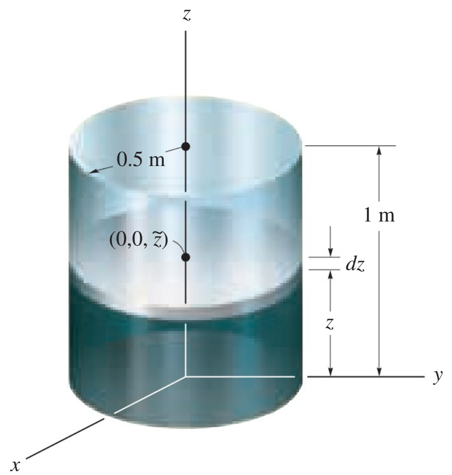


Fig. 9–15

SOLUTION

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Differential Element. A disk element of radius 0.5 m and thickness dz is chosen for integration, Fig. 9–15, since the *density of the entire element is constant* for a given value of z. The element is located along the z axis at the *arbitrary point* $(0, 0, z)$.

Volume and Moment Arm. The volume of the element is $dV = \pi(0.5)^2\, dz$, and its centroid is located at $\tilde{z} = z$.

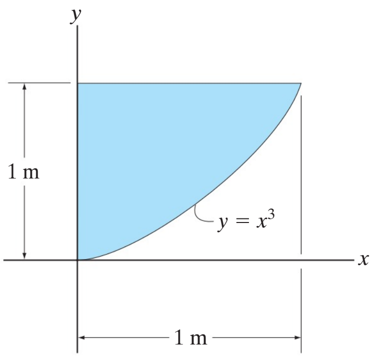
Integrations. Using the third of Eqs. 9–2 with $dm = \rho\, dV$ and integrating with respect to z, noting that $\rho = 200z$, we have

$$\bar{z} = \frac{\int_V \tilde{z}\rho\, dV}{\int_V \rho\, dV} = \frac{\int_0^{1\text{ m}} z(200z)\left[\pi(0.5)^2\, dz\right]}{\int_0^{1\text{ m}} (200z)\pi(0.5)^2\, dz}$$

$$= \frac{\int_0^{1\text{ m}} z^2\, dz}{\int_0^{1\text{ m}} z\, dz} = 0.667 \text{ m} \qquad \textit{Ans.}$$

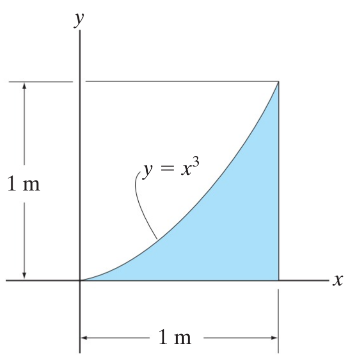
FUNDAMENTAL PROBLEMS

F9–1. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



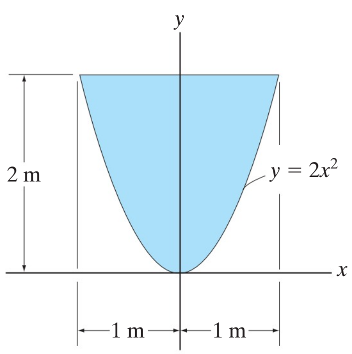
Prob. F9–1

F9–2. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



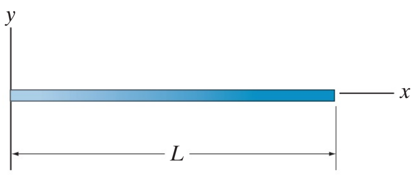
Prob. F9–2

F9–3. Determine the centroid $\bar{y}$ of the shaded area.



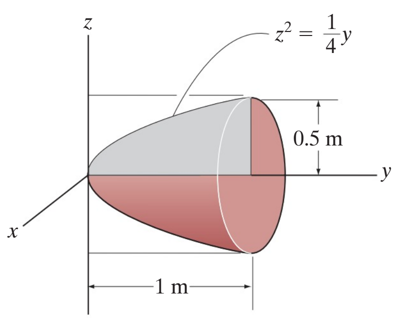
Prob. F9–3

F9–4. Locate the center of mass $\bar{x}$ of the straight rod if its mass per unit length is given by $m = m_0(1 + x^2/L^2)$.



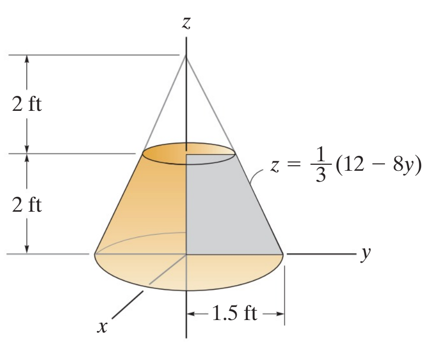
Prob. F9–4

F9–5. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.



Prob. F9–5

F9–6. Locate the centroid $\bar{z}$ of the homogeneous solid formed by revolving the shaded area about the z axis.



Prob. F9–6

9.2 Composite Bodies

A ***composite body*** consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. The method for doing this follows the same procedure outlined in Sec. 9.1. Formulas analogous to Eqs. 9–1 result; however, rather than account for an infinite number of differential weights, we have instead a finite number of weights. Therefore,

A stress analysis of this angle requires that the centroid of its cross-sectional area be located. (© Russell C. Hibbeler)

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W} \tag{9–6}$$

Here

$\bar{x}, \bar{y}, \bar{z}$ represent the coordinates of the center of gravity G of the composite body.

$\tilde{x}, \tilde{y}, \tilde{z}$ represent the coordinates of the center of gravity of each composite part of the body.

ΣW is the sum of the weights of all the composite parts of the body, or simply the total weight of the body.

When the body has a *constant density or specific weight*, the center of gravity *coincides* with the centroid of the body. The centroid for composite lines, areas, and volumes can be found using relations analogous to Eqs. 9–6; however, the W's are replaced by L's, A's, and V's, respectively. Centroids for common shapes of lines, areas, shells, and volumes that often make up a composite body are given in the table on the inside back cover.

In order to determine the force required to tip over this concrete barrier it is first necessary to determine the location of its center of gravity G. Due to symmetry, G will lie on the vertical axis of symmetry. (© Russell C. Hibbeler)

Procedure for Analysis

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by a line, area, or volume can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole*, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.

Summations.

- Determine $\bar{x}, \bar{y}, \bar{z}$ by applying the center of gravity equations, Eqs. 9–6, or the analogous centroid equations.
- If an object is *symmetrical* about an axis, the centroid of the object lies on this axis.

If desired, the calculations can be arranged in tabular form, as indicated in the following three examples.

The center of gravity of this water tank can be determined by dividing it into composite parts and applying Eqs. 9–6. (© Russell C. Hibbeler)

EXAMPLE 9.9

Locate the centroid of the wire shown in Fig. 9–16*a*.

SOLUTION

Composite Parts. The wire is divided into three segments as shown in Fig. 9–16*b*.

Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table on the inside back cover.

Summations. For convenience, the calculations can be tabulated as follows:

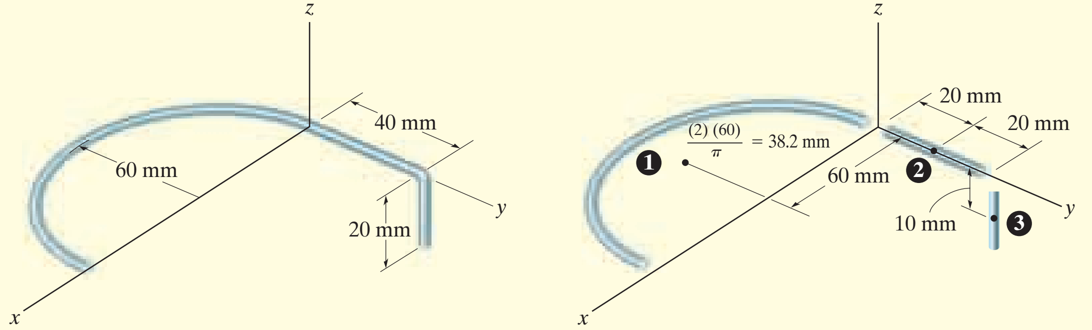
Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	-38.2	0	11 310	-7200	0
2	40	0	20	0	0	800	0
3	20	0	40	-10	0	800	-200
	$\Sigma L = 248.5$				$\Sigma\tilde{x}L = 11\,310$	$\Sigma\tilde{y}L = -5600$	$\Sigma\tilde{z}L = -200$

Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma\tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma\tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm} \qquad \textit{Ans.}$$



(a) (b)

Fig. 9–16

EXAMPLE 9.10

Locate the centroid of the plate area shown in Fig. 9–17*a*.

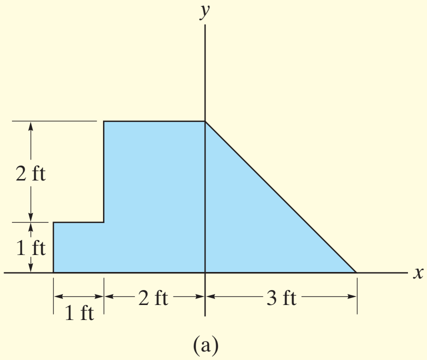
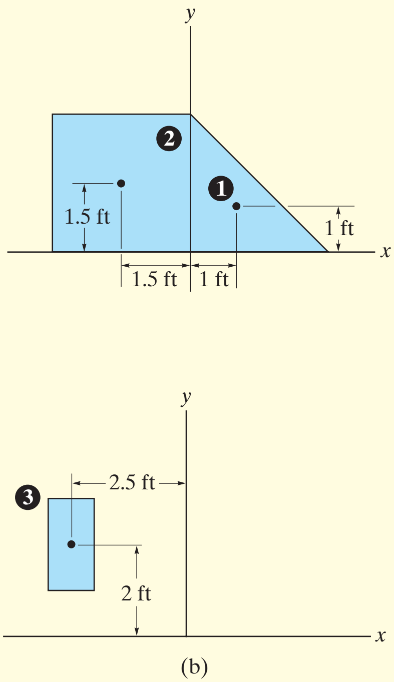


Fig. 9–17

SOLUTION

Composite Parts. The plate is divided into three segments as shown in Fig. 9–17*b*. Here the area of the small rectangle ③ is considered "negative" since it must be subtracted from the larger one ②.



Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of ② and ③ are *negative*.

Summations. Taking the data from Fig. 9–17*b*, the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	-1.5	1.5	-13.5	13.5
3	$-(2)(1) = -2$	-2.5	2	5	-4
	$\Sigma A = 11.5$			$\Sigma \tilde{x}A = -4$	$\Sigma \tilde{y}A = 14$

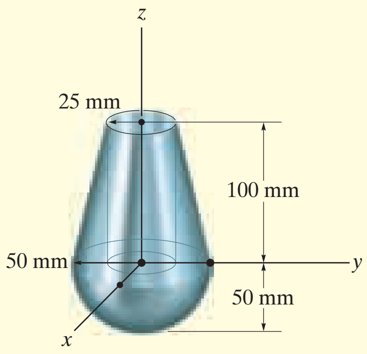
Thus,

$$\bar{x} = \frac{\Sigma \tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If these results are plotted in Fig. 9–17*a*, the location of point C seems reasonable.

EXAMPLE 9.11



(a)

Fig. 9–18

Locate the center of mass of the assembly shown in Fig. 9–18*a*. The conical frustum has a density of $\rho_c = 8\ \text{Mg/m}^3$, and the hemisphere has a density of $\rho_h = 4\ \text{Mg/m}^3$. There is a 25-mm-radius cylindrical hole in the center of the frustum.

SOLUTION

Composite Parts. The assembly can be thought of as consisting of four segments as shown in Fig. 9–18*b*. For the calculations, ③ and ④ must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape shown in Fig. 9–18*a*.

Moment Arm. Using the table on the inside back cover, the computations for the centroid $\tilde{z}$ of each piece are shown in the figure.

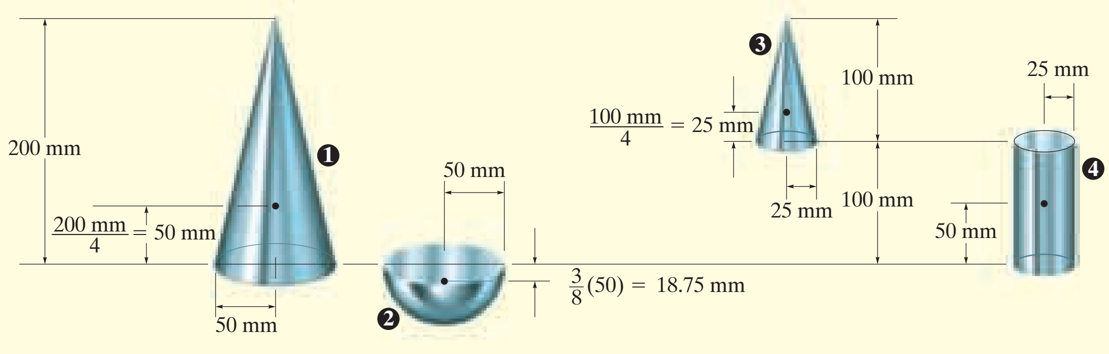
Summations. Because of *symmetry*, note that

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Since $W = mg$, and g is constant, the third of Eqs. 9–6 becomes $\bar{z} = \Sigma\tilde{z}m/\Sigma m$. The mass of each piece can be computed from $m = \rho V$ and used for the calculations. Also, $1\ \text{Mg/m}^3 = 10^{-6}\ \text{kg/mm}^3$, so that

Segment	m (kg)	$\tilde{z}$ (mm)	$\tilde{z}m$ (kg · mm)
1	$8(10^{-6})\left(\frac{1}{3}\right)\pi(50)^2(200) = 4.189$	50	209.440
2	$4(10^{-6})\left(\frac{2}{3}\right)\pi(50)^3 = 1.047$	−18.75	−19.635
3	$-8(10^{-6})\left(\frac{1}{3}\right)\pi(25)^2(100) = -0.524$	100 + 25 = 125	−65.450
4	$-8(10^{-6})\pi(25)^2(100) = -1.571$	50	−78.540
	$\Sigma m = 3.142$		$\Sigma\tilde{z}m = 45.815$

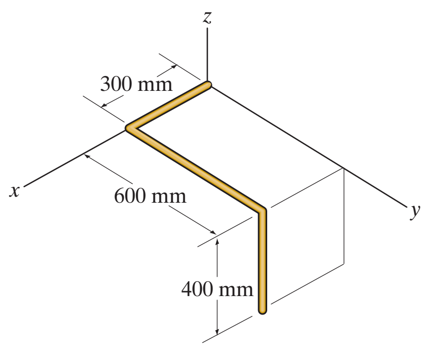
Thus, $$\bar{z} = \frac{\Sigma\tilde{z}m}{\Sigma m} = \frac{45.815}{3.142} = 14.6\ \text{mm} \qquad \textit{Ans.}$$



(b)

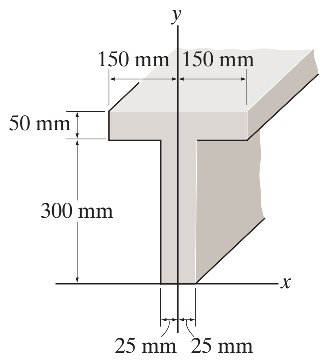
FUNDAMENTAL PROBLEMS

F9–7. Locate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the wire bent in the shape shown.



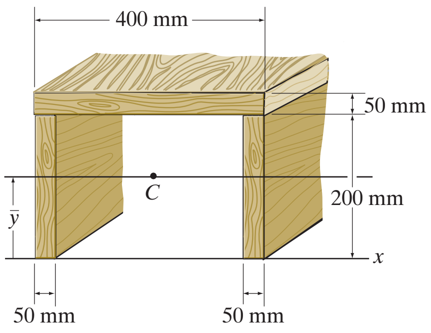
Prob. F9–7

F9–8. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



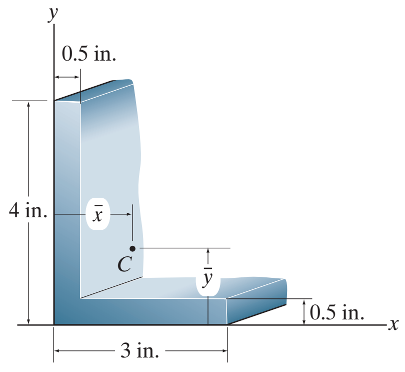
Prob. F9–8

F9–9. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



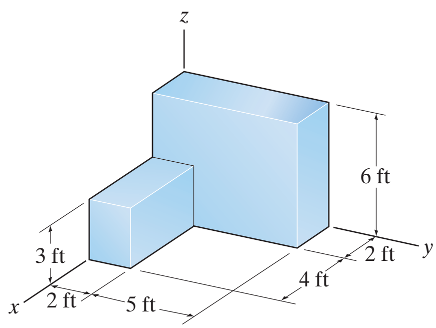
Prob. F9–9

F9–10. Locate the centroid $(\bar{x}, \bar{y})$ of the cross-sectional area.



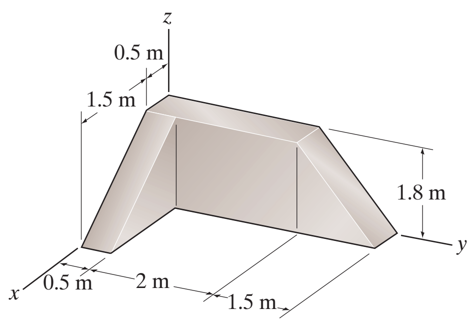
Prob. F9–10

F9–11. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



Prob. F9–11

F9–12. Determine the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



Prob. F9–12

*9.3 Theorems of Pappus and Guldinus

The two ***theorems of Pappus and Guldinus*** are used to find the surface area and volume of any body of revolution. They were first developed by Pappus of Alexandria during the fourth century A.D. and then restated at a later time by the Swiss mathematician Paul Guldin or Guldinus (1577–1643).

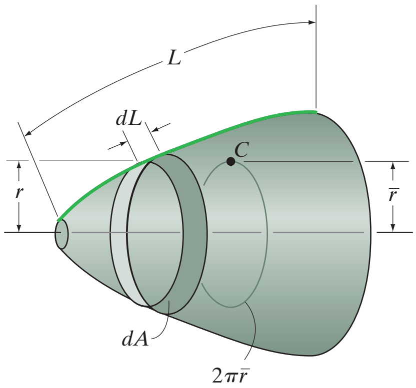


Fig. 9–19

Surface Area. If we revolve a *plane curve* about an axis that does not intersect the curve we will generate a *surface area of revolution*. For example, the surface area in Fig. 9–19 is formed by revolving the curve of length L about the horizontal axis. To determine this surface area, we will first consider the differential line element of length dL. If this element is revolved 2π radians about the axis, a ring having a surface area of $dA = 2\pi r\,dL$ will be generated. Thus, the surface area of the entire body is $A = 2\pi \int r\,dL$. Since $\int r\,dL = \bar{r}L$ (Eq. 9–5), then $A = 2\pi\bar{r}L$. If the curve is revolved only through an angle θ (radians), then

$$A = \theta \bar{r} L \tag{9–7}$$

where

A = surface area of revolution

θ = angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating curve

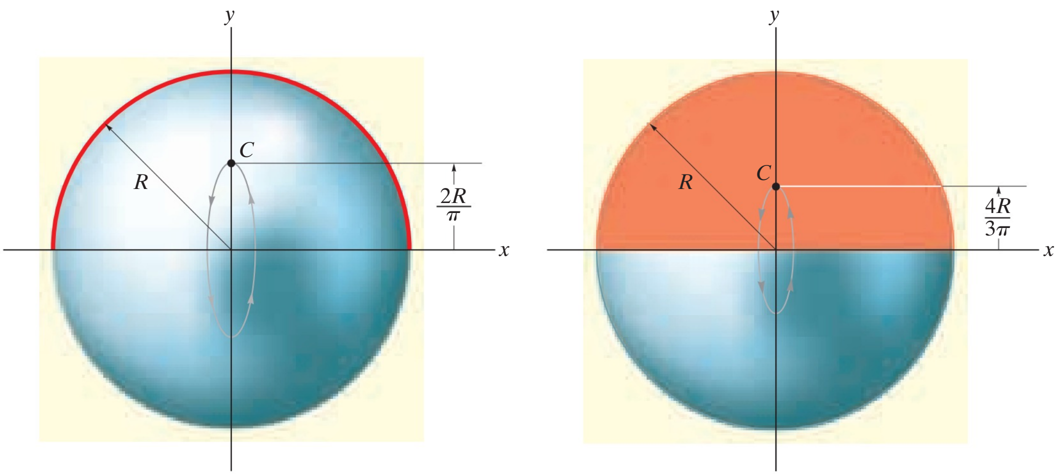
L = length of the generating curve

Therefore the first theorem of Pappus and Guldinus states that *the area of a surface of revolution equals the product of the length of the generating curve and the distance traveled by the centroid of the curve in generating the surface area.*

The amount of material used on this storage building can be estimated by using the first theorem of Pappus and Guldinus to determine its surface area. (© Russell C. Hibbeler)

EXAMPLE 9.12

Show that the surface area of a sphere is $A = 4\pi R^2$ and its volume is $V = \frac{4}{3}\pi R^3$.



(a) (b)

Fig. 9–21

SOLUTION

Surface Area. The surface area of the sphere in Fig. 9–21*a* is generated by revolving a semicircular *arc* about the *x* axis. Using the table on the inside back cover, it is seen that the centroid of this arc is located at a distance $\bar{r} = 2R/\pi$ from the axis of revolution (*x* axis). Since the centroid moves through an angle of $\theta = 2\pi$ rad to generate the sphere, then applying Eq. 9–7 we have

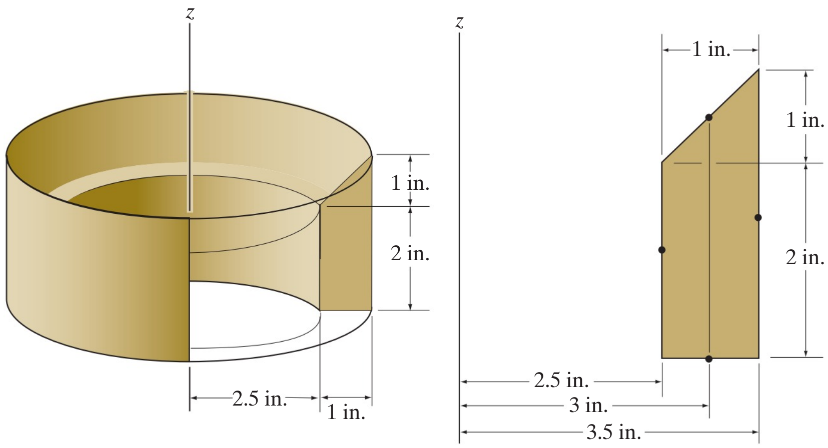
$$A = \theta\bar{r}L; \qquad A = 2\pi\left(\frac{2R}{\pi}\right)\pi R = 4\pi R^2 \qquad \textit{Ans.}$$

Volume. The volume of the sphere is generated by revolving the semicircular *area* in Fig. 9–21*b* about the *x* axis. Using the table on the inside back cover to locate the centroid of the area, i.e., $\bar{r} = 4R/3\pi$, and applying Eq. 9–8, we have

$$V = \theta\bar{r}A; \qquad V = 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right) = \frac{4}{3}\pi R^3 \qquad \textit{Ans.}$$

EXAMPLE 9.13

Determine the surface area and volume of the full solid in Fig. 9–22*a*.



(a) (b)

Fig. 9–22

SOLUTION

Surface Area. The surface area is generated by revolving the four line segments shown in Fig. 9–22*b* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–7 yields

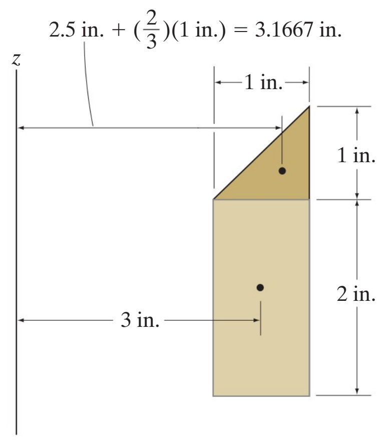
$$A = 2\pi\Sigma\bar{r}L = 2\pi\left[(2.5\text{ in.})(2\text{ in.}) + (3\text{ in.})\left(\sqrt{(1\text{ in.})^2 + (1\text{ in.})^2}\right) + (3.5\text{ in.})(3\text{ in.}) + (3\text{ in.})(1\text{ in.})\right]$$

$$= 143\text{ in}^2 \qquad \textit{Ans.}$$

Volume. The volume of the solid is generated by revolving the two area segments shown in Fig. 9–22*c* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–10, we have

$$V = 2\pi\Sigma\bar{r}A = 2\pi\left\{(3.1667\text{ in.})\left[\frac{1}{2}(1\text{ in.})(1\text{ in.})\right] + (3\text{ in.})[(2\text{ in.})(1\text{ in.})]\right\}$$

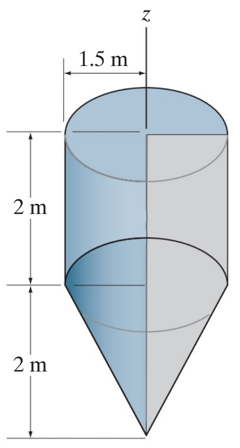
$$= 47.6\text{ in}^3 \qquad \textit{Ans.}$$



(c)

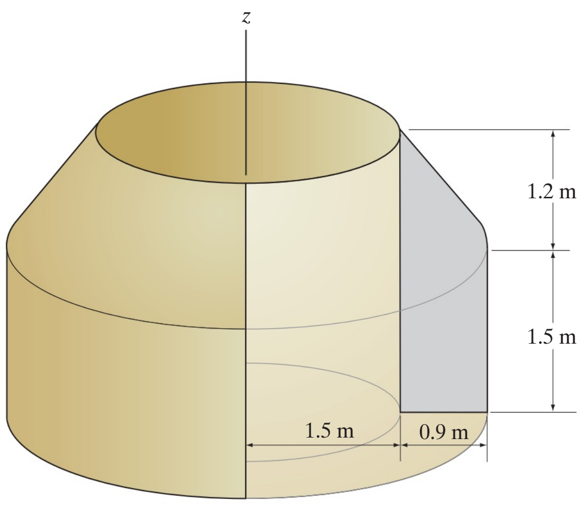
FUNDAMENTAL PROBLEMS

F9–13. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



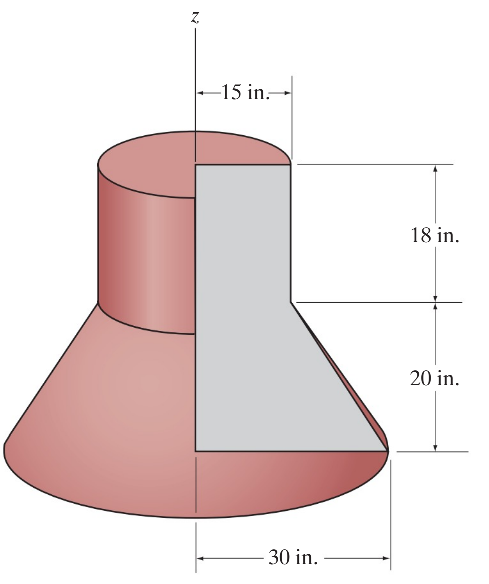
Prob. F9–13

F9–14. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



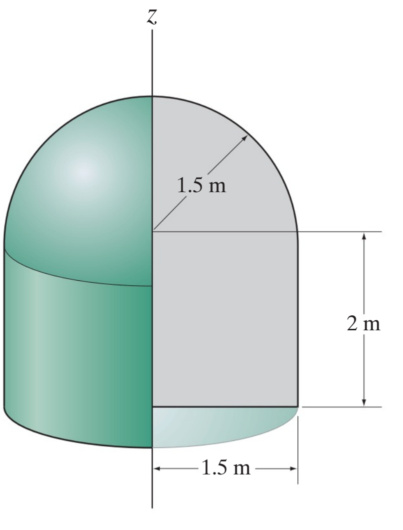
Prob. F9–14

F9–15. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



Prob. F9–15

F9–16. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



Prob. F9–16

Lecture 9

Moments of Inertia

CHAPTER OBJECTIVES

- To develop a method for determining the moment of inertia for an area.
- To introduce the product of inertia and show how to determine the maximum and minimum moments of inertia for an area.
- To discuss the mass moment of inertia.

10.1 Definition of Moments of Inertia for Areas

Whenever a distributed load acts perpendicular to an area and its intensity varies linearly, the calculation of the moment of the loading about an axis will involve an integral of the form $\int y^2 dA$. For example, consider the plate in Fig. 10–1, which is submerged in a fluid and subjected to the pressure p. As discussed in Sec. 9.5, this pressure varies linearly with depth, such that $p = \gamma y$, where γ is the specific weight of the fluid. Thus, the force acting on the differential area dA of the plate is $dF = p\,dA = (\gamma y)dA$. The *moment* of this force about the x axis is therefore $dM = y\,dF = \gamma y^2 dA$, and so integrating dM over the entire area of the plate yields $M = \gamma \int y^2 dA$. The integral $\int y^2 dA$ is sometimes referred to as the "second moment" of the area about an axis (the x axis), but more often it is called the ***moment of inertia of the area***. The word "inertia" is used here since the formulation is similar to the mass moment of inertia, $\int y^2 dm$, which is a dynamical property described in Sec. 10.8. Although for an area this integral has no physical meaning, it often arises in formulas used in fluid mechanics, mechanics of materials, structural mechanics, and mechanical design, and so the engineer needs to be familiar with the methods used to determine the moment of inertia.

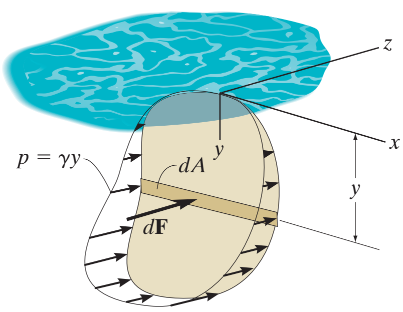


Fig. 10–1

Moment of Inertia. By definition, the moments of inertia of a differential area dA about the x and y axes are $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively, Fig. 10–2. For the entire area A the ***moments of inertia*** are determined by integration; i.e.,

$$I_x = \int_A y^2\,dA \qquad (10\text{–}1)$$
$$I_y = \int_A x^2\,dA$$

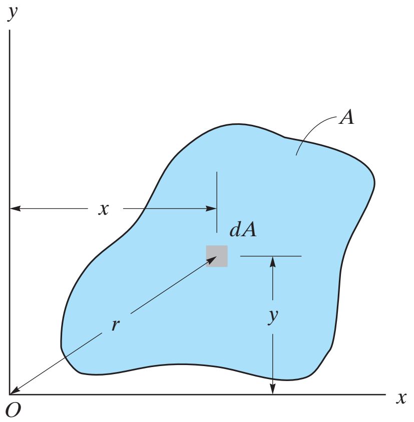


Fig. 10–2

We can also formulate this quantity for dA about the "pole" O or z axis, Fig. 10–2. This is referred to as the ***polar moment of inertia***. It is defined as $dJ_O = r^2\,dA$, where r is the perpendicular distance from the pole (z axis) to the element dA. For the entire area the *polar moment of inertia* is

$$J_O = \int_A r^2\,dA = I_x + I_y \qquad (10\text{–}2)$$

This relation between J_O and I_x, I_y is possible since $r^2 = x^2 + y^2$, Fig. 10–2.

From the above formulations it is seen that I_x, I_y, and J_O will *always* be *positive* since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4, mm^4, or ft^4, $in.^4$.

10.2 Parallel-Axis Theorem for an Area

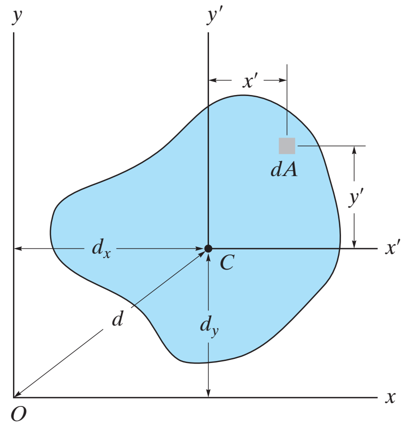


Fig. 10–3

The ***parallel-axis theorem*** can be used to find the moment of inertia of an area about *any axis* that is parallel to an axis passing through the centroid and about which the moment of inertia is known. To develop this theorem, we will consider finding the moment of inertia of the shaded area shown in Fig. 10–3 about the x axis. To start, we choose a differential element dA located at an arbitrary distance y' from the *centroidal* x' axis. If the distance between the parallel x and x' axis is d_y, then the moment of inertia of dA about the x axis is $dI_x = (y' + d_y)^2\,dA$. For the entire area,

$$I_x = \int_A (y' + d_y)^2\,dA$$
$$= \int_A y'^2\,dA + 2d_y \int_A y'\,dA + d_y^2 \int_A dA$$

The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$. The second integral is zero since the x' axis passes through the area's centroid C; i.e., $\int y'\,dA = \bar{y}'\int dA = 0$ since $\bar{y}' = 0$. Since the third integral represents the total area A, the final result is therefore

$$I_x = \bar{I}_{x'} + Ad_y^2 \tag{10–3}$$

A similar expression can be written for I_y; i.e.,

$$I_y = \bar{I}_{y'} + Ad_x^2 \tag{10–4}$$

And finally, for the polar moment of inertia, since $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$ and $d^2 = d_x^2 + d_y^2$, we have

$$J_O = \bar{J}_C + Ad^2 \tag{10–5}$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes*.

In order to predict the strength and deflection of this beam, it is necessary to calculate the moment of inertia of the beam's cross-sectional area. (© Russell C. Hibbeler)

10.3 Radius of Gyration of an Area

The ***radius of gyration*** of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are *known*, the radii of gyration are determined from the formulas

$$k_x = \sqrt{\frac{I_x}{A}}$$
$$k_y = \sqrt{\frac{I_y}{A}} \tag{10–6}$$
$$k_O = \sqrt{\frac{J_O}{A}}$$

The form of these equations is easily remembered since it is similar to that for finding the moment of inertia for a differential area about an axis. For example, $I_x = k_x^2 A$; whereas for a differential area, $dI_x = y^2\,dA$.

Important Points

- The moment of inertia is a geometric property of an area that is used to determine the strength of a structural member or the location of a resultant pressure force acting on a plate submerged in a fluid. It is sometimes referred to as the second moment of the area about an axis, because the distance from the axis to each area element is squared.
- If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

Procedure for Analysis

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the *arbitrary point* (x, y).

Case 1.

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is computed. This situation occurs when the rectangular element shown in Fig. 10–4*a* is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 10–4*b*. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

Case 2.

- The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is computed; however, Eq. 10–1 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 10–4*a* is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y. See Examples 10.2 and 10.3.

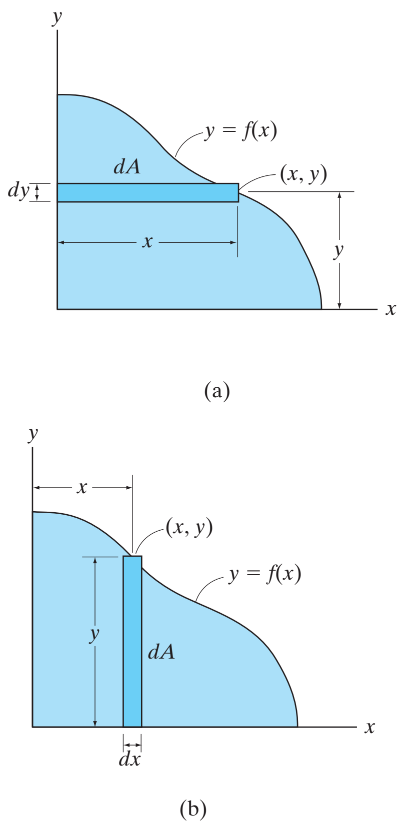


Fig. 10–4

EXAMPLE 10.1

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x'–y' plane and passing through the centroid C.

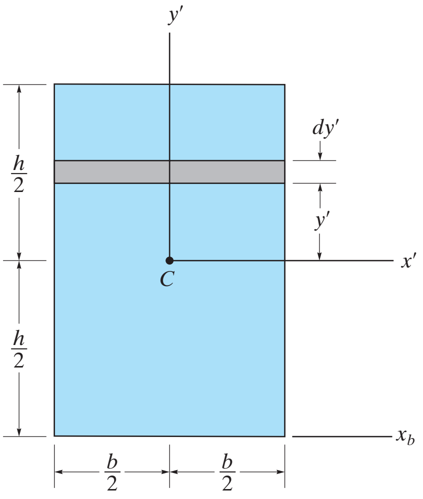


Fig. 10–5

SOLUTION (CASE 1)

Part (a). The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

$$I_{x_b} = \bar{I}_{x'} + A d_y^2$$

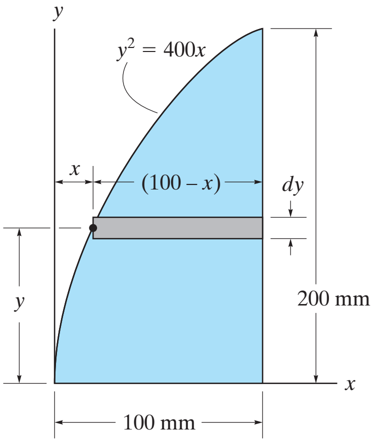
$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \qquad \textit{Ans.}$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10–2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad \textit{Ans.}$$



(a)

Determine the moment of inertia for the shaded area shown in Fig. 10–6*a* about the x axis.

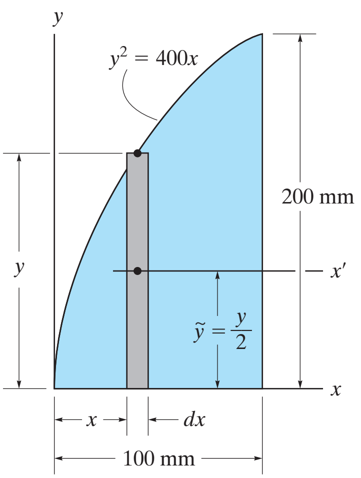
SOLUTION I (CASE 1)

A differential element of area that is *parallel* to the x axis, as shown in Fig. 10–6*a*, is chosen for integration. Since this element has a thickness dy and intersects the curve at the *arbitrary point* (x, y), its area is $dA = (100 - x)\,dy$. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

$$I_x = \int_A y^2\,dA = \int_0^{200\text{ mm}} y^2(100 - x)\,dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right)dy = \int_0^{200\text{ mm}}\left(100y^2 - \frac{y^4}{400}\right)dy$$

$$= 107(10^6)\text{ mm}^4$$ *Ans.*



(b)

Fig. 10–6

SOLUTION II (CASE 2)

A differential element *parallel* to the y axis, as shown in Fig. 10–6*b*, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the x axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 10.1. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 10–6*b*, $b = dx$ and $h = y$, and thus $d\bar{I}_{x'} = \frac{1}{12}dx\,y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the x axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\,\tilde{y}^2 = \frac{1}{12}dx\,y^3 + y\,dx\left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\,dx$$

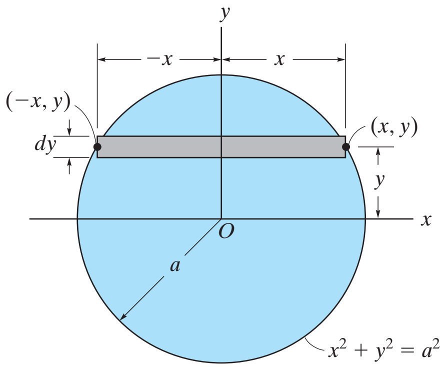
(This result can also be concluded from part (b) of Example 10.1.) Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

$$I_x = \int dI_x = \int_0^{100\text{ mm}} \frac{1}{3}y^3\,dx = \int_0^{100\text{ mm}} \frac{1}{3}(400x)^{3/2}\,dx$$

$$= 107(10^6)\text{ mm}^4$$ *Ans.*

EXAMPLE 10.3

Determine the moment of inertia with respect to the x axis for the circular area shown in Fig. 10–7a.



(a)

SOLUTION I (CASE 1)

Using the differential element shown in Fig. 10–7a, since $dA = 2x\,dy$, we have

$$I_x = \int_A y^2\,dA = \int_A y^2(2x)\,dy$$

$$= \int_{-a}^{a} y^2\left(2\sqrt{a^2 - y^2}\right) dy = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

SOLUTION II (CASE 2)

When the differential element shown in Fig. 10–7b is chosen, the centroid for the element happens to lie on the x axis, and since $\bar{I}_{x'} = \frac{1}{12}bh^3$ for a rectangle, we have

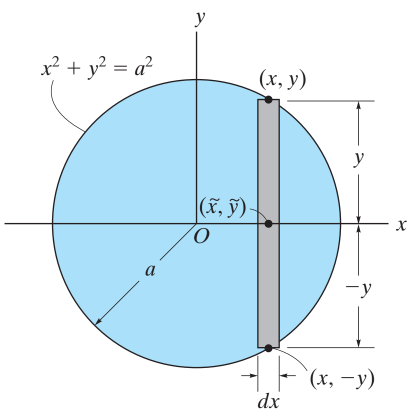
$$dI_x = \frac{1}{12}dx(2y)^3$$

$$= \frac{2}{3}y^3\,dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a} \frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

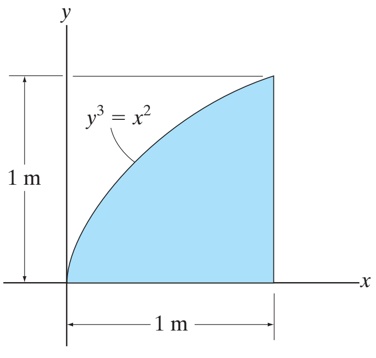
NOTE: By comparison, Solution I requires much less computation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.



(b)

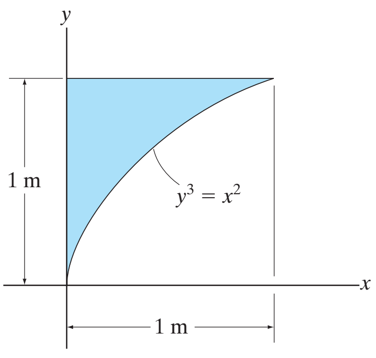
Fig. 10–7

F10–1. Determine the moment of inertia of the shaded area about the x axis.



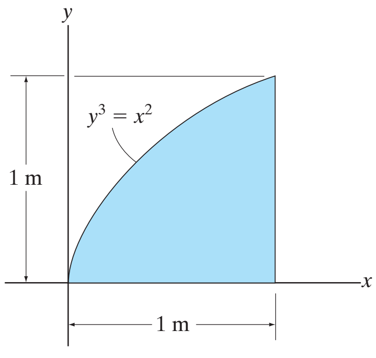
Prob. F10–1

F10–2. Determine the moment of inertia of the shaded area about the x axis.



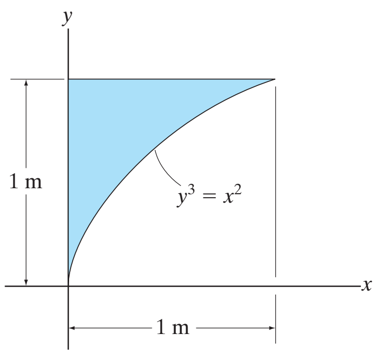
Prob. F10–2

F10–3. Determine the moment of inertia of the shaded area about the y axis.



Prob. F10–3

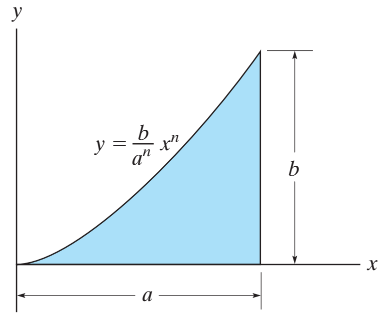
F10–4. Determine the moment of inertia of the shaded area about the y axis.



Prob. F10–4

10–1. Determine the moment of inertia about the x axis.

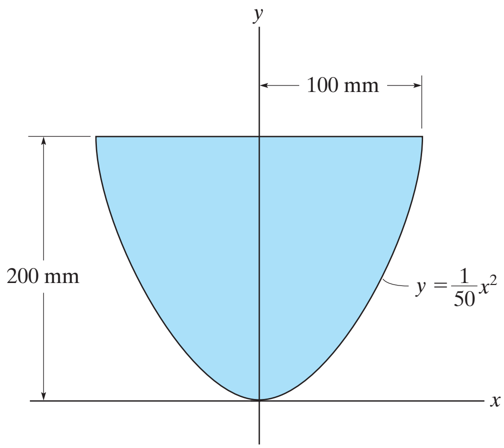
10–2. Determine the moment of inertia about the y axis.



Probs. 10–1/2

10–3. Determine the moment of inertia for the shaded area about the x axis.

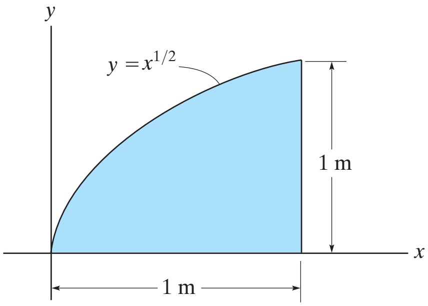
***10–4.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–3/4

10–5. Determine the moment of inertia for the shaded area about the x axis.

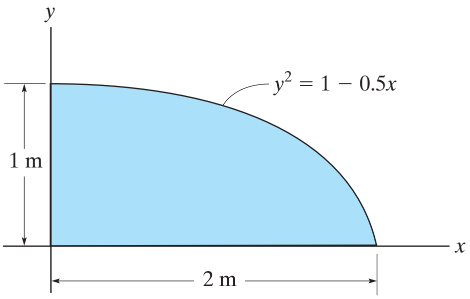
10–6. Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–5/6

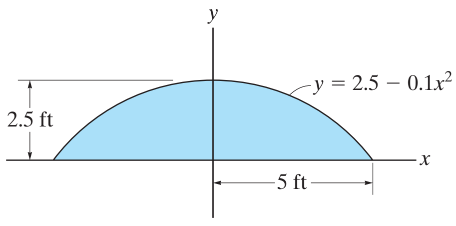
10–7. Determine the moment of inertia for the shaded area about the x axis.

***10–8.** Determine the moment of inertia for the shaded area about the y axis.



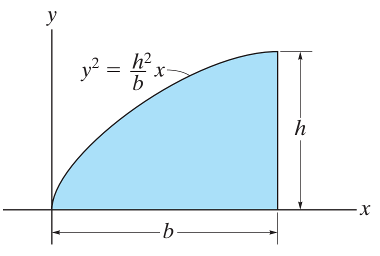
Probs. 10–7/8

10–9. Determine the moment of inertia of the area about the x axis. Solve the problem in two ways, using rectangular differential elements: (a) having a thickness dx and (b) having a thickness of dy.



Prob. 10–9

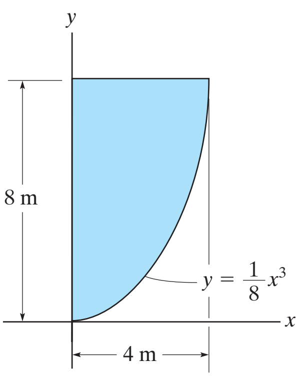
10–10. Determine the moment of inertia of the area about the x axis.



Prob. 10–10

10–11. Determine the moment of inertia for the shaded area about the x axis.

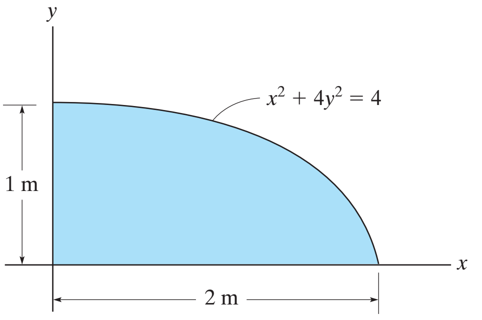
***10–12.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–11/12

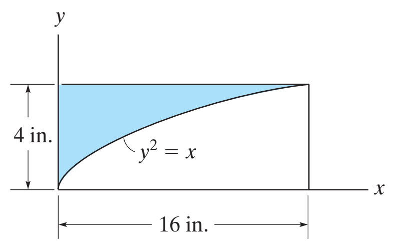
10–13. Determine the moment of inertia about the x axis.

10–14. Determine the moment of inertia about the y axis.



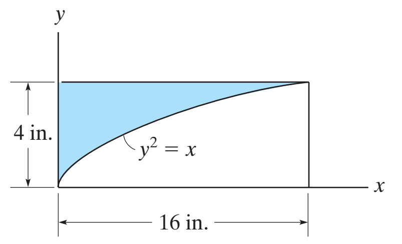
Probs. 10–13/14

10–15. Determine the moment of inertia for the shaded area about the x axis.



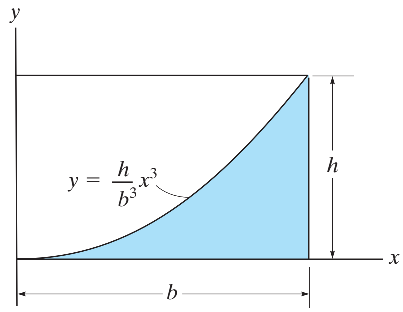
Prob. 10–15

***10–16.** Determine the moment of inertia for the shaded area about the y axis.



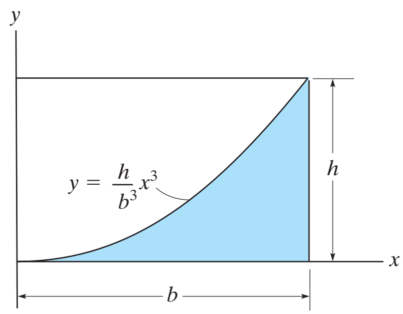
Prob. 10–16

10–17. Determine the moment of inertia for the shaded area about the x axis.



Prob. 10–17

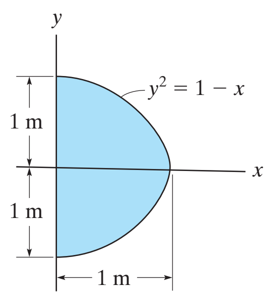
10–18. Determine the moment of inertia for the shaded area about the y axis.



Prob. 10–18

10–19. Determine the moment of inertia for the shaded area about the x axis.

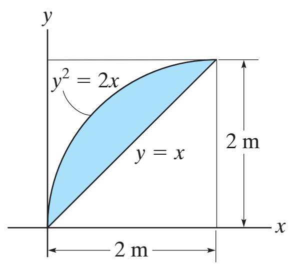
***10–20.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–19/20

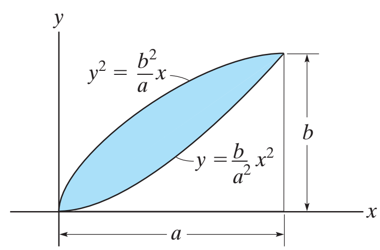
10–21. Determine the moment of inertia for the shaded area about the x axis.

10–22. Determine the moment of inertia for the shaded area about the y axis.



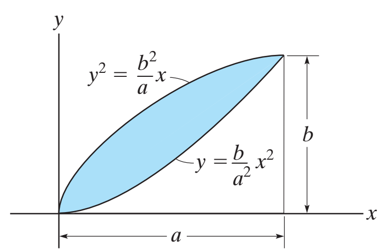
Probs. 10–21/22

10–23. Determine the moment of inertia for the shaded area about the x axis.



Prob. 10–23

***10–24.** Determine the moment of inertia for the shaded area about the y axis.



Prob. 10–24

10.4 Moments of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the *algebraic sum* of the moments of inertia of all its parts.

Procedure for Analysis

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the reference axis. For the calculation of $\bar{I}$, use the table on the inside back cover.

Summation.

- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.
- If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.

For design or analysis of this T-beam, engineers must be able to locate the centroid of its cross-sectional area, and then find the moment of inertia of this area about the centroidal axis. (© Russell C. Hibbeler)

EXAMPLE 10.4

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the *x* axis.

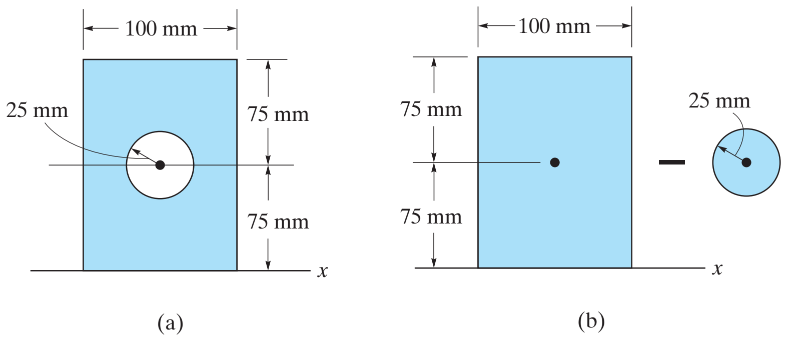


Fig. 10–8

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the *x* axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$; $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\text{ mm}^4$$

Rectangle

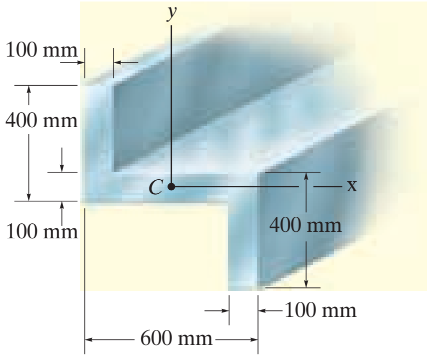
$$I_x = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\text{ mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$



(a)

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9*a* about the *x* and *y* centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas *A*, *B*, and *D* shown in Fig. 10–9*b*. For the calculation, the centroid of each of these rectangles is located in the figure.

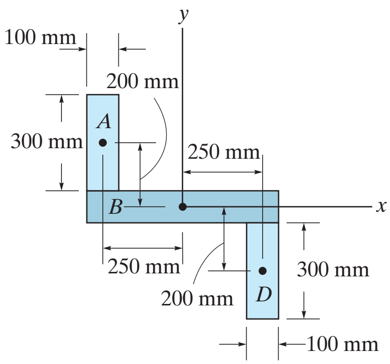
Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles *A* and *D*, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + A d_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\ \text{mm}^4$$
$$I_y = \bar{I}_{y'} + A d_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\ \text{mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$



(b)

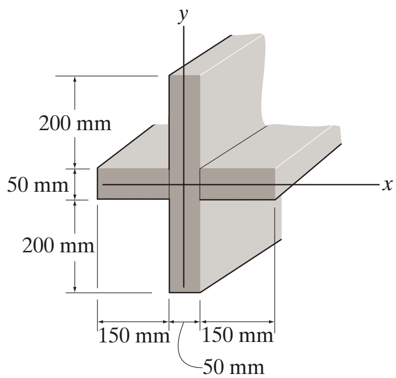
Fig. 10–9

Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

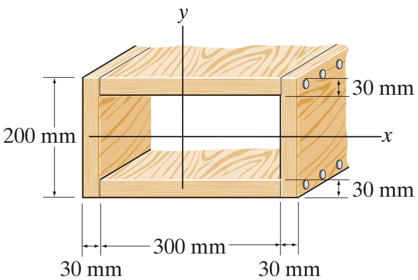
FUNDAMENTAL PROBLEMS

F10–5. Determine the moment of inertia of the beam's cross-sectional area about the centroidal *x* and *y* axes.



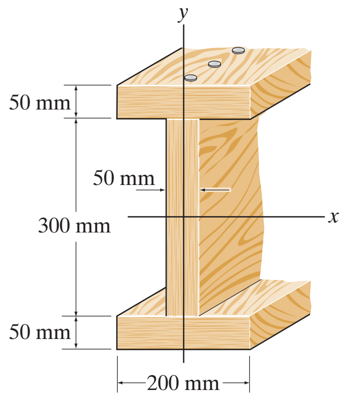
Prob. F10–5

F10–6. Determine the moment of inertia of the beam's cross-sectional area about the centroidal *x* and *y* axes.



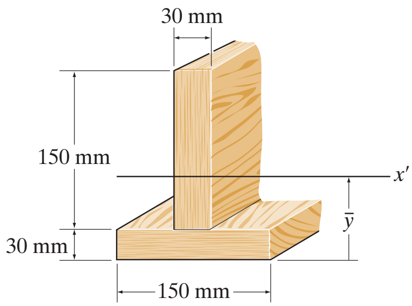
Prob. F10–6

F10–7. Determine the moment of inertia of the cross-sectional area of the channel with respect to the *y* axis.



Prob. F10–7

F10–8. Determine the moment of inertia of the cross-sectional area of the T-beam with respect to the x' axis passing through the centroid of the cross section.



Prob. F10–8

Lecture 10

10.8 Mass Moment of Inertia

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration. Since it is used in dynamics to study rotational motion, methods for its calculation will now be discussed.*

Consider the rigid body shown in Fig. 10–21. We define the *mass moment of inertia* of the body about the z axis as

$$I = \int_m r^2\, dm \qquad (10\text{–}12)$$

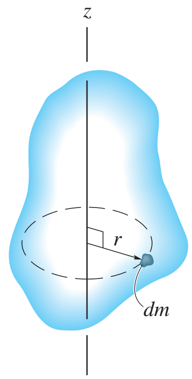


Fig. 10–21

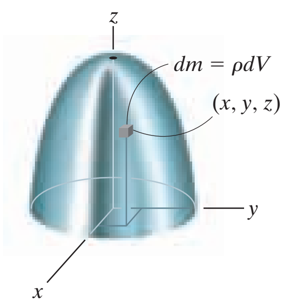
Here r is the perpendicular distance from the axis to the arbitrary element dm. Since the formulation involves r, the value of I is *unique* for each axis about which it is computed. The axis which is generally chosen, however, passes through the body's mass center G. Common units used for its measurement are $\text{kg} \cdot \text{m}^2$ or $\text{slug} \cdot \text{ft}^2$.

If the body consists of material having a density ρ, then $dm = \rho\, dV$, Fig. 10–22*a*. Substituting this into Eq. 10–12, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho\, dV \qquad (10\text{–}13)$$

For most applications, ρ will be a *constant*, and so this term may be factored out of the integral, and the integration is then purely a function of geometry.

$$I = \rho \int_V r^2\, dV \qquad (10\text{–}14)$$



(a)

Fig. 10–22

*Another property of the body, which measures the symmetry of the body's mass with respect to a coordinate system, is the mass product of inertia. This property most often applies to the three-dimensional motion of a body and is discussed in *Engineering Mechanics: Dynamics* (Chapter 21).

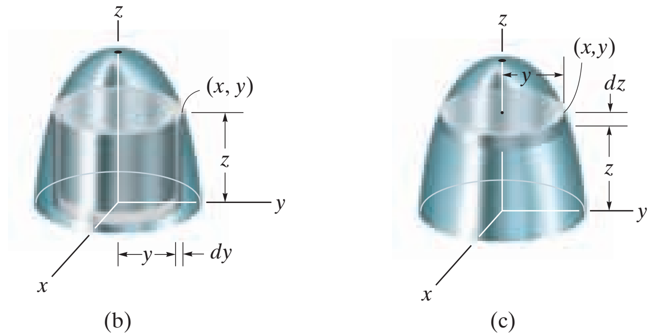


Fig. 10–22 (cont'd)

Procedure for Analysis

If a body is symmetrical with respect to an axis, as in Fig. 10–22, then its mass moment of inertia about the axis can be determined by using a single integration. Shell and disk elements are used for this purpose.

Shell Element.

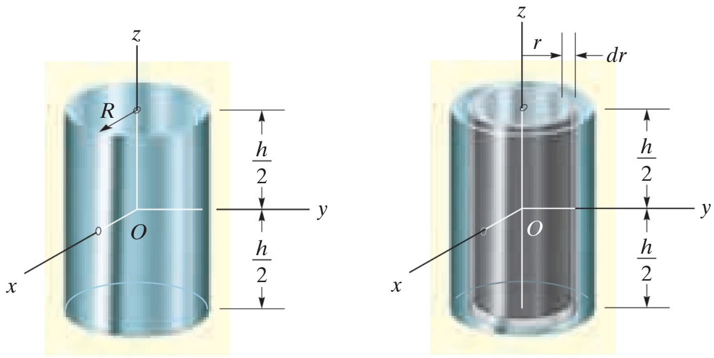
- If a *shell element* having a height z, radius y, and thickness dy is chosen for integration, Fig. 10–22*b*, then its volume is $dV = (2\pi y)(z)\,dy$.
- This element can be used in Eq. 10–13 or 10–14 for determining the moment of inertia I_z of the body about the z axis since the *entire element*, due to its "thinness," lies at the *same* perpendicular distance $r = y$ from the z axis (see Example 10.10).

Disk Element.

- If a disk element having a radius y and a thickness dz is chosen for integration, Fig. 10–22*c*, then its volume is $dV = (\pi y^2)\,dz$.
- In this case the element is *finite* in the radial direction, and consequently its points *do not* all lie at the *same radial distance* r from the z axis. As a result, Eqs. 10–13 or 10–14 *cannot* be used to determine I_z. Instead, to perform the integration using this element, it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result (see Example 10.11).

EXAMPLE 10.10

Determine the mass moment of inertia of the cylinder shown in Fig. 10–23*a* about the z axis. The density of the material, ρ, is constant.



(a) (b)

Fig. 10–23

SOLUTION

Shell Element. This problem will be solved using the *shell element* in Fig. 10–23*b* and thus only a single integration is required. The volume of the element is $dV = (2\pi r)(h)\,dr$, and so its mass is $dm = \rho\,dV = \rho(2\pi hr\,dr)$. Since the *entire element* lies at the same distance r from the z axis, the moment of inertia *of the element* is

$$dI_z = r^2\,dm = \rho 2\pi h r^3\,dr$$

Integrating over the entire cylinder yields

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2} R^4 h$$

Since the mass of the cylinder is

$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

then

$$I_z = \frac{1}{2} m R^2 \qquad \textit{Ans.}$$

10

If the density of the solid in Fig. 10–24*a* is 5 slug/ft^3, determine the mass moment of inertia about the y axis.

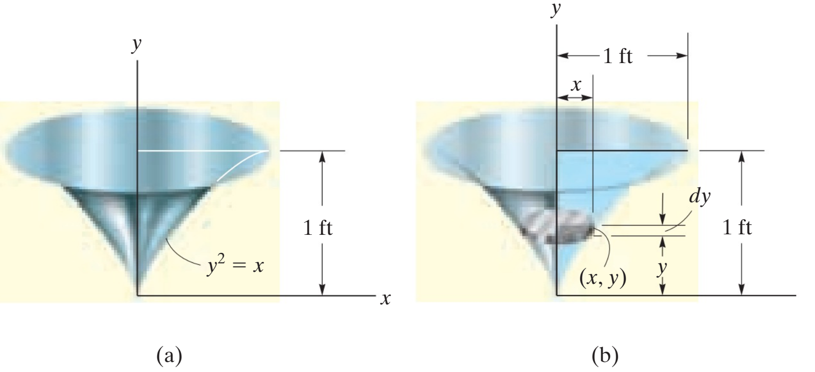


Fig. 10–24

SOLUTION

Disk Element. The moment of inertia will be determined using this *disk element*, as shown in Fig. 10–24*b*. Here the element intersects the curve at the arbitrary point (x, y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all points on the element are *not* located at the same distance from the y axis, it is still possible to determine the moment of inertia dI_y *of the element* about the y axis. In the previous example it was shown that the moment of inertia of a homogeneous cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height of the cylinder is not involved in this formula, we can also use this result for a disk. Thus, for the disk element in Fig. 10–24*b*, we have

$$dI_y = \frac{1}{2}(dm)x^2 = \frac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

Substituting $x = y^2$, $\rho = 5$ slug/ft^3, and integrating with respect to y, from $y = 0$ to $y = 1$ ft, yields the moment of inertia for the entire solid.

$$I_y = \frac{5\pi}{2}\int_0^{1\text{ ft}} x^4\, dy = \frac{5\pi}{2}\int_0^{1\text{ ft}} y^8\, dy = 0.873 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

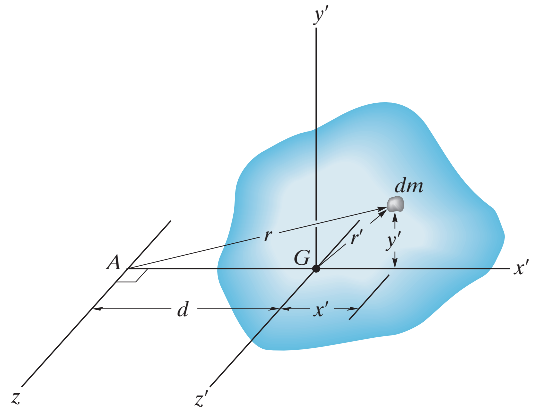


Fig. 10–25

Parallel-Axis Theorem. If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the *parallel-axis theorem*. To derive this theorem, consider the body shown in Fig. 10–25. The z' axis passes through the mass center G, whereas the corresponding *parallel z axis* lies at a constant distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, the moment of inertia of the body about the z axis is

$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral is equal to *zero*, since the z' axis passes through the body's mass center, i.e., $\int x'\, dm = \bar{x} \int dm = 0$ since $\bar{x} = 0$. Finally, the third integral is the total mass m of the body. Hence, the moment of inertia about the z axis becomes

$$I = I_G + md^2 \tag{10–15}$$

where

I_G = moment of inertia about the z' axis passing through the mass center G

m = mass of the body

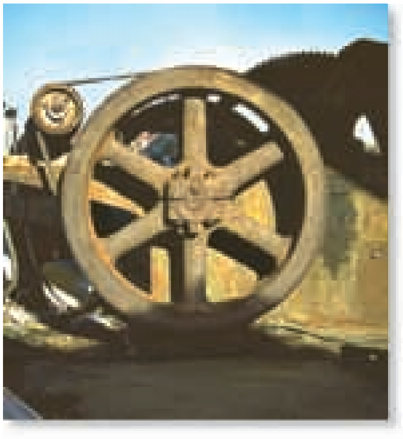
d = distance between the parallel axes

Radius of Gyration. Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the *radius of gyration, k*. This value has units of length, and when it and the body's mass m are known, the moment of inertia can be determined from the equation

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}} \tag{10–16}$$

Note the *similarity* between the definition of k in this formula and r in the equation $dI = r^2\,dm$, which defines the moment of inertia of a differential element of mass dm of the body about an axis.

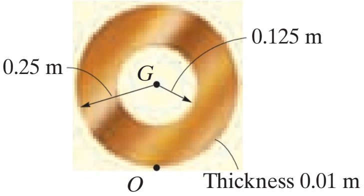
Composite Bodies. If a body is constructed from a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis z can be determined by adding algebraically the moments of inertia of all the composite shapes calculated about the same axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been included within another part—as in the case of a "hole" subtracted from a solid plate. Also, the parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the z axis. For calculations, a table of some simple shapes is given on the inside back cover.



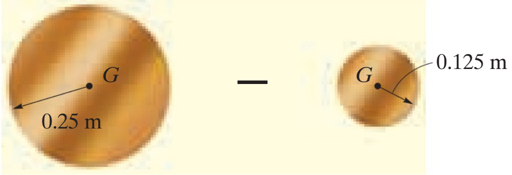
This flywheel, which operates a metal cutter, has a large moment of inertia about its center. Once it begins rotating it is difficult to stop it and therefore a uniform motion can be effectively transferred to the cutting blade. (© Russell C. Hibbeler)

EXAMPLE 10.12

If the plate shown in Fig. 10–26*a* has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its mass moment of inertia about an axis perpendicular to the page and passing through the pin at O.



(a)



(b)

Fig. 10–26

SOLUTION

The plate consists of two composite parts, the 250-mm-radius disk *minus* a 125-mm-radius disk, Fig. 10–26*b*. The moment of inertia about O can be determined by finding the moment of inertia of each of these parts about O and then *algebraically* adding the results. The calculations are performed by using the parallel-axis theorem in conjunction with the mass moment of inertia formula for a circular disk, $I_G = \frac{1}{2}mr^2$, as found on the inside back cover.

Disk. The moment of inertia of a disk about an axis perpendicular to the plane of the disk and passing through G is $I_G = \frac{1}{2}mr^2$. The mass center of both disks is 0.25 m from point O. Thus,

$$m_d = \rho_d V_d = 8000 \text{ kg/m}^3 \, [\pi(0.25 \text{ m})^2(0.01 \text{ m})] = 15.71 \text{ kg}$$

$$\begin{aligned}(I_O)_d &= \tfrac{1}{2}m_d r_d^2 + m_d d^2 \\ &= \tfrac{1}{2}(15.71 \text{ kg})(0.25 \text{ m})^2 + (15.71 \text{ kg})(0.25 \text{ m})^2 \\ &= 1.473 \text{ kg}\cdot\text{m}^2\end{aligned}$$

Hole. For the smaller disk (hole), we have

$$m_h = \rho_h V_h = 8000 \text{ kg/m}^3 \, [\pi(0.125 \text{ m})^2(0.01 \text{ m})] = 3.93 \text{ kg}$$

$$\begin{aligned}(I_O)_h &= \tfrac{1}{2}m_h r_h^2 + m_h d^2 \\ &= \tfrac{1}{2}(3.93 \text{ kg})(0.125 \text{ m})^2 + (3.93 \text{ kg})(0.25 \text{ m})^2 \\ &= 0.276 \text{ kg}\cdot\text{m}^2\end{aligned}$$

The moment of inertia of the plate about the pin is therefore

$$\begin{aligned}I_O &= (I_O)_d - (I_O)_h \\ &= 1.473 \text{ kg}\cdot\text{m}^2 - 0.276 \text{ kg}\cdot\text{m}^2 \\ &= 1.20 \text{ kg}\cdot\text{m}^2\end{aligned}$$

Ans.

EXAMPLE 10.13

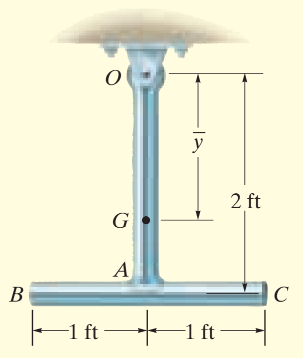


Fig. 10–27

The pendulum in Fig. 10–27 consists of two thin rods each having a weight of 10 lb. Determine the pendulum's mass moment of inertia about an axis passing through (a) the pin at O, and (b) the mass center G of the pendulum.

SOLUTION

Part (a). Using the table on the inside back cover, the moment of inertia of rod OA about an axis perpendicular to the page and passing through the end point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 = 0.414 \text{ slug}\cdot\text{ft}^2$$

Realize that this same value may be determined using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem; i.e.,

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(1 \text{ ft})^2$$
$$= 0.414 \text{ slug}\cdot\text{ft}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(2 \text{ ft})^2$$
$$= 1.346 \text{ slug}\cdot\text{ft}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.414 + 1.346 = 1.76 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to the pin at O. Assuming this distance to be $\bar{y}$, Fig. 10–27, and using the formula for determining the mass center, we have

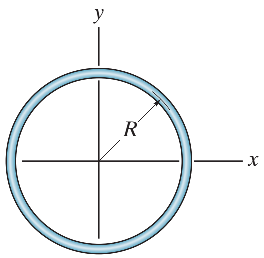
$$\bar{y} = \frac{\Sigma\tilde{y}m}{\Sigma m} = \frac{1(10/32.2) + 2(10/32.2)}{(10/32.2) + (10/32.2)} = 1.50 \text{ ft}$$

The moment of inertia I_G may be computed in the same manner as I_O, which requires successive applications of the parallel-axis theorem in order to transfer the moments of inertia of rods OA and BC to G. A more direct solution, however, involves applying the parallel-axis theorem using the result for I_O determined above; i.e.,

$$I_O = I_G + md^2; \qquad 1.76 \text{ slug}\cdot\text{ft}^2 = I_G + \left(\frac{20 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(1.50 \text{ ft})^2$$
$$I_G = 0.362 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

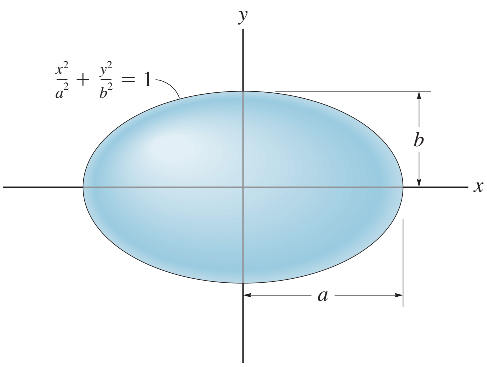
PROBLEMS

***10–84.** Determine the moment of inertia of the thin ring about the z axis. The ring has a mass m.



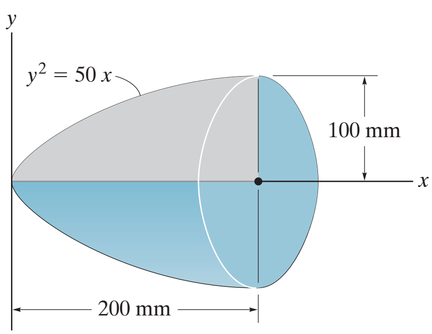
Prob. 10–84

10–85. Determine the moment of inertia of the ellipsoid with respect to the x axis and express the result in terms of the mass m of the ellipsoid. The material has a constant density ρ.



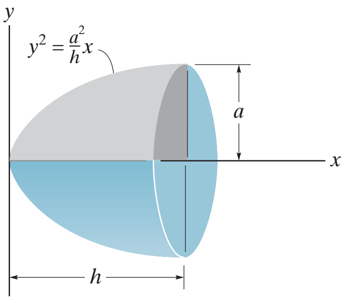
Prob. 10–85

10–86. Determine the radius of gyration k_x of the paraboloid. The density of the material is $\rho = 5\ \mathrm{Mg/m^3}$.



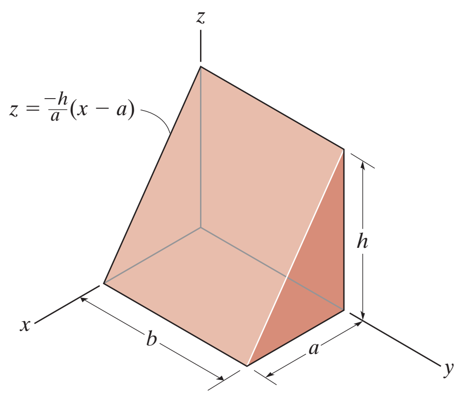
Prob. 10–86

10–87. The paraboloid is formed by revolving the shaded area around the x axis. Determine the moment of inertia about the x axis and express the result in terms of the total mass m of the paraboloid. The material has a constant density ρ.



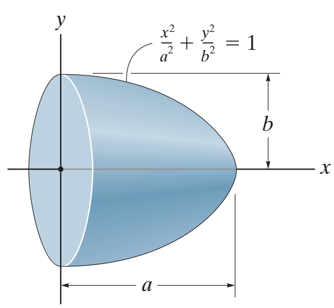
Prob. 10–87

***10–88.** Determine the moment of inertia of the homogenous triangular prism with respect to the y axis. Express the result in terms of the mass m of the prism. *Hint*: For integration, use thin plate elements parallel to the x–y plane having a thickness of dz.



Prob. 10–88

10–89. Determine the moment of inertia of the semiellipsoid with respect to the x axis and express the result in terms of the mass m of the semiellipsoid. The material has a constant density ρ.



Prob. 10–89

Lecture 11

Virtual Work

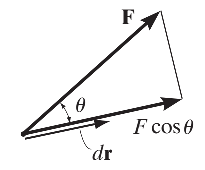
CHAPTER OBJECTIVES

- To introduce the principle of virtual work and show how it applies to finding the equilibrium configuration of a system of pin-connected members.
- To establish the potential-energy function and use the potential-energy method to investigate the type of equilibrium or stability of a rigid body or system of pin-connected members.

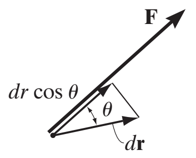
11.1 Definition of Work

The principle of virtual work was proposed by the Swiss mathematician Jean Bernoulli in the eighteenth century. It provides an alternative method for solving problems involving the equilibrium of a particle, a rigid body, or a system of connected rigid bodies. Before we discuss this principle, however, we must first define the work produced by a force and by a couple moment.

Work of a Force. A force does work when it undergoes a displacement in the direction of its line of action. Consider, for example, the force $\mathbf{F}$ in Fig. 11–1*a* that undergoes a differential displacement $d\mathbf{r}$. If θ is the angle between the force and the displacement, then the component of $\mathbf{F}$ in



(a)



(b)

Fig. 11–1

the direction of the displacement is $F\cos\theta$. And so the work produced by **F** is

$$dU = F\,dr\cos\theta$$

Notice that this expression is also the product of the force F and the component of displacement in the direction of the force, $dr\cos\theta$, Fig. 11–1*b*. If we use the definition of the dot product (Eq. 2–11) the work can also be written as

$$dU = \mathbf{F}\cdot d\mathbf{r}$$

As the above equations indicate, work is a ***scalar***, and like other scalar quantities, it has a magnitude that can either be *positive* or *negative*.

In the SI system, the unit of work is a ***joule*** (J), which is the work produced by a 1-N force that displaces through a distance of 1 m in the direction of the force (1 J = 1 N · m). The unit of work in the FPS system is the foot-pound (ft · lb), which is the work produced by a 1-lb force that displaces through a distance of 1 ft in the direction of the force.

The moment of a force has this same combination of units; however, the concepts of moment and work are in no way related. A moment is a vector quantity, whereas work is a scalar.

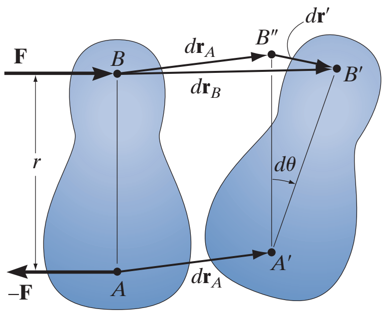


Fig. 11–2

Work of a Couple Moment. The rotation of a couple moment also produces work. Consider the rigid body in Fig. 11–2, which is acted upon by the couple forces **F** and –**F** that produce a couple moment **M** having a magnitude $M = Fr$. When the body undergoes the differential displacement shown, points A and B move $d\mathbf{r}_A$ and $d\mathbf{r}_B$ to their final positions A' and B', respectively. Since $d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}'$, this movement can be thought of as a *translation* $d\mathbf{r}_A$, where A and B move to A' and B'', and a *rotation* about A', where the body rotates through the angle $d\theta$ about A. The couple forces do no work during the translation $d\mathbf{r}_A$ because each force undergoes the same amount of displacement in opposite directions, thus canceling out the work. During rotation, however, **F** is displaced $dr' = r\,d\theta$, and so it does work $dU = F\,dr' = F\,r\,d\theta$. Since $M = Fr$, the work of the couple moment **M** is therefore

$$dU = M\,d\theta$$

If **M** and $d\theta$ have the same sense, the work is *positive*; however, if they have the opposite sense, the work will be *negative*.

Virtual Work. The definitions of the work of a force and a couple have been presented in terms of *actual movements* expressed by differential displacements having magnitudes of dr and $d\theta$. Consider now an *imaginary* or ***virtual movement*** of a body in static equilibrium, which indicates a displacement or rotation that is *assumed* and *does not actually exist*. These movements are first-order differential quantities and will be denoted by the symbols δr and $\delta\theta$ (delta r and delta θ), respectively. The *virtual work* done by a force having a virtual displacement δr is

$$\delta U = F\cos\theta\,\delta r \qquad (11\text{–}1)$$

Similarly, when a couple undergoes a virtual rotation $\delta\theta$ in the plane of the couple forces, the *virtual work* is

$$\delta U = M\,\delta\theta \qquad (11\text{–}2)$$

11.2 Principle of Virtual Work

The ***principle of virtual*** work states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

$$\delta U = 0 \qquad (11\text{–}3)$$

For example, consider the free-body diagram of the particle (ball) that rests on the floor, Fig. 11–3. If we "imagine" the ball to be displaced downwards a virtual amount δy, then the weight does positive virtual work, $W\,\delta y$, and the normal force does negative virtual work, $-N\,\delta y$. For equilibrium the total virtual work must be zero, so that $\delta U = W\,\delta y - N\,\delta y = (W - N)\,\delta y = 0$. Since $\delta y \neq 0$, then $N = W$ as required by applying $\Sigma F_y = 0$.

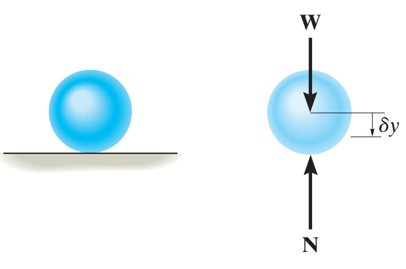


Fig. 11–3

In a similar manner, we can also apply the virtual-work equation $\delta U = 0$ to a rigid body subjected to a coplanar force system. Here, separate virtual translations in the x and y directions, and a virtual rotation about an axis perpendicular to the x–y plane that passes through an arbitrary point O, will correspond to the three equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_O = 0$. When writing these equations, it is *not necessary* to include the work done by the *internal forces* acting within the body since a rigid body *does not deform* when subjected to an external loading, and furthermore, when the body moves through a virtual displacement, the internal forces occur in equal but opposite collinear pairs, so that the corresponding work done by each pair of forces will cancel.

To demonstrate an application, consider the simply supported beam in Fig. 11–4*a*. When the beam is given a virtual rotation $\delta\theta$ about point B, Fig. 11–4*b*, the only forces that do work are **P** and $\mathbf{A}_y$. Since $\delta y = l\,\delta\theta$ and $\delta y' = (l/2)\,\delta\theta$, the virtual work equation for this case is $\delta U = A_y(l\,\delta\theta) - P(l/2)\,\delta\theta = (A_y l - Pl/2)\,\delta\theta = 0$. Since $\delta\theta \neq 0$, then $A_y = P/2$. Excluding $\delta\theta$, notice that the terms in parentheses actually represent the application of $\Sigma M_B = 0$.

As seen from the above two examples, no added advantage is gained by solving particle and rigid-body equilibrium problems using the principle of virtual work. This is because for each application of the virtual-work equation, the virtual displacement, common to every term, factors out, leaving an equation that could have been obtained in a more *direct manner* by simply applying an equation of equilibrium.

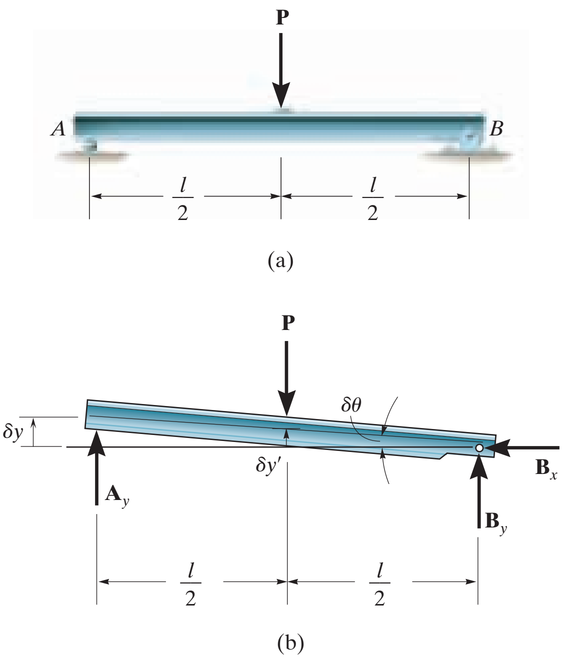


Fig. 11–4

11.3 Principle of Virtual Work for a System of Connected Rigid Bodies

The method of virtual work is particularly effective for solving equilibrium problems that involve a system of several *connected* rigid bodies, such as the ones shown in Fig. 11–5.

Each of these systems is said to have only one degree of freedom since the arrangement of the links can be completely specified using only one coordinate θ. In other words, with this single coordinate and the length of the members, we can locate the position of the forces **F** and **P**.

In this text, we will only consider the application of the principle of virtual work to systems containing one degree of freedom.* Because they are less complicated, they will serve as a way to approach the solution of more complex problems involving systems with many degrees of freedom. The procedure for solving problems involving a system of frictionless connected rigid bodies follows.

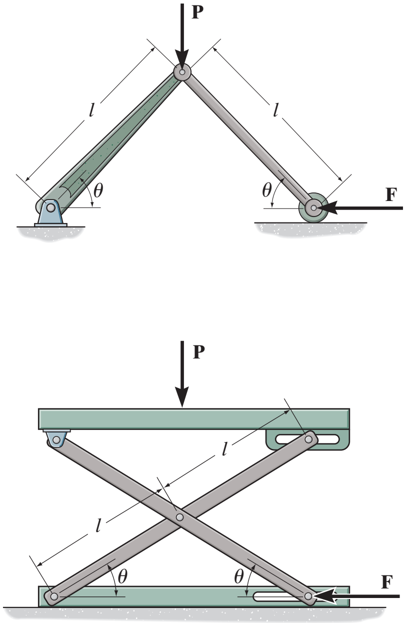
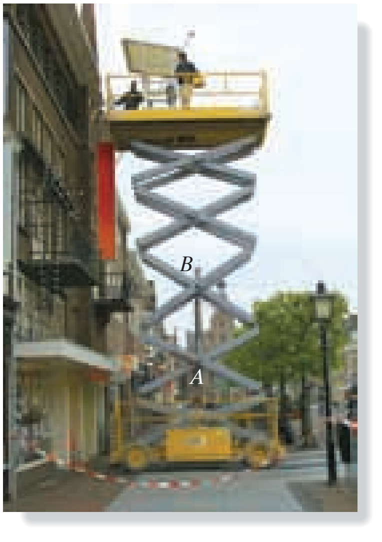


Fig. 11–5

Important Points

- A force does work when it moves through a displacement in the direction of the force. A couple moment does work when it moves through a collinear rotation. Specifically, positive work is done when the force or couple moment and its displacement have the same sense of direction.
- The principle of virtual work is generally used to determine the equilibrium configuration for a system of multiple connected members.
- A virtual displacement is imaginary; i.e., it does not really happen. It is a differential displacement that is given in the positive direction of a position coordinate.
- Forces or couple moments that do not virtually displace do no virtual work.



This scissors lift has one degree of freedom. Without the need for dismembering the mechanism, the force in the hydraulic cylinder AB required to provide the lift can be determined *directly* by using the principle of virtual work. (© Russell C. Hibbeler)

*This method of applying the principle of virtual work is sometimes called the *method of virtual displacements* because a virtual displacement is applied, resulting in the calculation of a real force. Although it is not used here, we can also apply the principle of virtual work as a *method of virtual forces*. This method is often used to apply a virtual force and then determine the displacements of points on deformable bodies. See R. C. Hibbeler, *Mechanics of Materials*, 8th edition, Pearson/Prentice Hall, 2011.

Procedure for Analysis

Free-Body Diagram.

- Draw the free-body diagram of the entire system of connected bodies and define the *coordinate* q.
- Sketch the "deflected position" of the system on the free-body diagram when the system undergoes a *positive virtual* displacement δq.

Virtual Displacements.

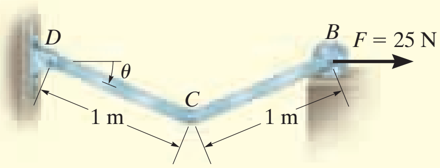
- Indicate *position coordinates* s, each measured from a *fixed point* on the free-body diagram. These coordinates are directed to the forces that do work.
- Each of these coordinate axes should be *parallel* to the line of action of the force to which it is directed, so that the virtual work along the coordinate axis can be calculated.
- Relate each of the position coordinates s to the coordinate q; then *differentiate* these expressions in order to express each virtual displacement δs in terms of δq.

Virtual-Work Equation.

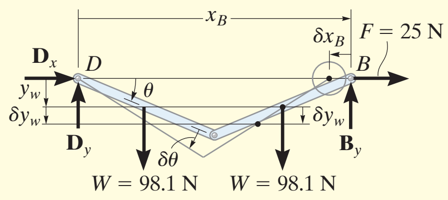
- Write the *virtual-work equation* for the system assuming that, whether possible or not, each position coordinate s undergoes a *positive* virtual displacement δs. If a force or couple moment is in the same direction as the positive virtual displacement, the work is positive. Otherwise, it is negative.
- Express the work of *each* force and couple moment in the equation in terms of δq.
- Factor out this common displacement from all the terms, and solve for the unknown force, couple moment, or equilibrium position q.

EXAMPLE 11.1

Determine the angle θ for equilibrium of the two-member linkage shown in Fig. 11–6*a*. Each member has a mass of 10 kg.



(a)



(b)

Fig. 11–6

SOLUTION

Free-Body Diagram. The system has only one degree of freedom since the location of both links can be specified by the single coordinate, $(q =)\ \theta$. As shown on the free-body diagram in Fig. 11–6*b*, when θ has a *positive* (clockwise) virtual rotation $\delta\theta$, only the force **F** and the two 98.1-N weights do work. (The reactive forces $\mathbf{D}_x$ and $\mathbf{D}_y$ are fixed, and $\mathbf{B}_y$ does not displace along its line of action.)

Virtual Displacements. If the origin of coordinates is established at the *fixed* pin support *D*, then the position of **F** and **W** can be specified by the *position coordinates* x_B and y_w. In order to determine the work, note that, as required, these coordinates are parallel to the lines of action of their associated forces. Expressing these position coordinates in terms of θ and taking the derivatives yields

$$x_B = 2(1 \cos\theta)\text{ m} \qquad \delta x_B = -2\sin\theta\,\delta\theta\text{ m} \qquad (1)$$

$$y_w = \tfrac{1}{2}(1 \sin\theta)\text{ m} \qquad \delta y_w = 0.5\cos\theta\,\delta\theta\text{ m} \qquad (2)$$

It is seen by the *signs* of these equations, and indicated in Fig. 11–6*b*, that an *increase* in θ (i.e., $\delta\theta$) causes a *decrease* in x_B and an *increase* in y_w.

Virtual-Work Equation. If the virtual displacements δx_B and δy_w were *both positive*, then the forces **W** and **F** would do positive work since the forces and their corresponding displacements would have the same sense. Hence, the virtual-work equation for the displacement $\delta\theta$ is

$$\delta U = 0; \qquad W\,\delta y_w + W\,\delta y_w + F\,\delta x_B = 0 \qquad (3)$$

Substituting Eqs. 1 and 2 into Eq. 3 in order to relate the virtual displacements to the common virtual displacement $\delta\theta$ yields

$$98.1(0.5\cos\theta\,\delta\theta) + 98.1(0.5\cos\theta\,\delta\theta) + 25(-2\sin\theta\,\delta\theta) = 0$$

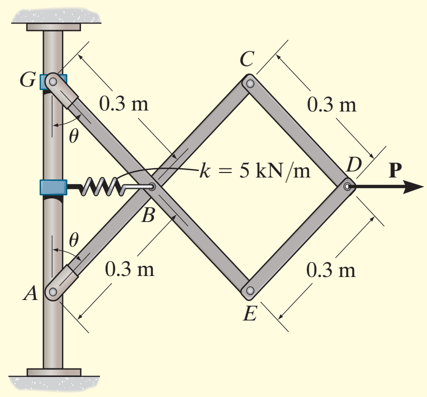
Notice that the "negative work" done by **F** (force in the opposite sense to displacement) has actually been *accounted for* in the above equation by the "negative sign" of Eq. 1. Factoring out the *common displacement* $\delta\theta$ and solving for θ, noting that $\delta\theta \neq 0$, yields

$$(98.1\cos\theta - 50\sin\theta)\,\delta\theta = 0$$

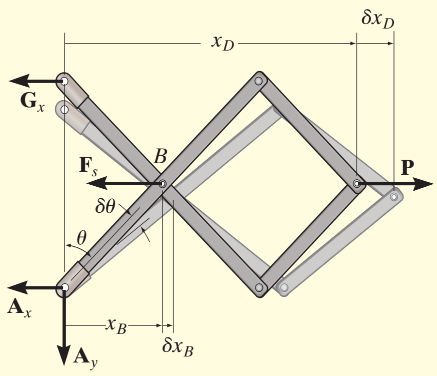
$$\theta = \tan^{-1}\frac{98.1}{50} = 63.0^\circ \qquad \textit{Ans.}$$

NOTE: If this problem had been solved using the equations of equilibrium, it would be necessary to dismember the links and apply three scalar equations to *each* link. The principle of virtual work, by means of calculus, has eliminated this task so that the answer is obtained directly.

EXAMPLE 11.2



(a)



(b)

Fig. 11–7

Determine the required force P in Fig. 11–7*a* needed to maintain equilibrium of the scissors linkage when $\theta = 60^\circ$. The spring is unstretched when $\theta = 30^\circ$. Neglect the mass of the links.

SOLUTION

Free-Body Diagram. Only $\mathbf{F}_s$ and $\mathbf{P}$ do work when θ undergoes a *positive* virtual displacement $\delta\theta$, Fig. 11–7*b*. For the arbitrary position θ, the spring is stretched $(0.3 \text{ m}) \sin\theta - (0.3 \text{ m}) \sin 30^\circ$, so that

$$F_s = ks = 5000 \text{ N/m}\,[(0.3 \text{ m}) \sin\theta - (0.3 \text{ m}) \sin 30^\circ]$$
$$= (1500 \sin\theta - 750) \text{ N}$$

Virtual Displacements. The position coordinates, x_B and x_D, measured from the *fixed point* A, are used to locate $\mathbf{F}_s$ and $\mathbf{P}$. These coordinates are parallel to the line of action of their corresponding forces. Expressing x_B and x_D in terms of the angle θ using trigonometry,

$$x_B = (0.3 \text{ m}) \sin\theta$$
$$x_D = 3[(0.3 \text{ m}) \sin\theta] = (0.9 \text{ m}) \sin\theta$$

Differentiating, we obtain the virtual displacements of points B and D.

$$\delta x_B = 0.3 \cos\theta\,\delta\theta \qquad (1)$$
$$\delta x_D = 0.9 \cos\theta\,\delta\theta \qquad (2)$$

Virtual-Work Equation. Force $\mathbf{P}$ does positive work since it acts in the positive sense of its virtual displacement. The spring force $\mathbf{F}_s$ does negative work since it acts opposite to its positive virtual displacement. Thus, the virtual-work equation becomes

$$\delta U = 0; \qquad -F_s\,\delta x_B + P\delta x_D = 0$$
$$-[1500 \sin\theta - 750]\,(0.3 \cos\theta\,\delta\theta) + P\,(0.9 \cos\theta\,\delta\theta) = 0$$
$$[0.9P + 225 - 450 \sin\theta] \cos\theta\,\delta\theta = 0$$

Since $\cos\theta\,\delta\theta \neq 0$, then this equation requires

$$P = 500 \sin\theta - 250$$

When $\theta = 60^\circ$,

$$P = 500 \sin 60^\circ - 250 = 183 \text{ N} \qquad \textit{Ans.}$$

EXAMPLE 11.3

If the box in Fig. 11–8*a* has a mass of 10 kg, determine the couple moment M needed to maintain equilibrium when $\theta = 60°$. Neglect the mass of the members.

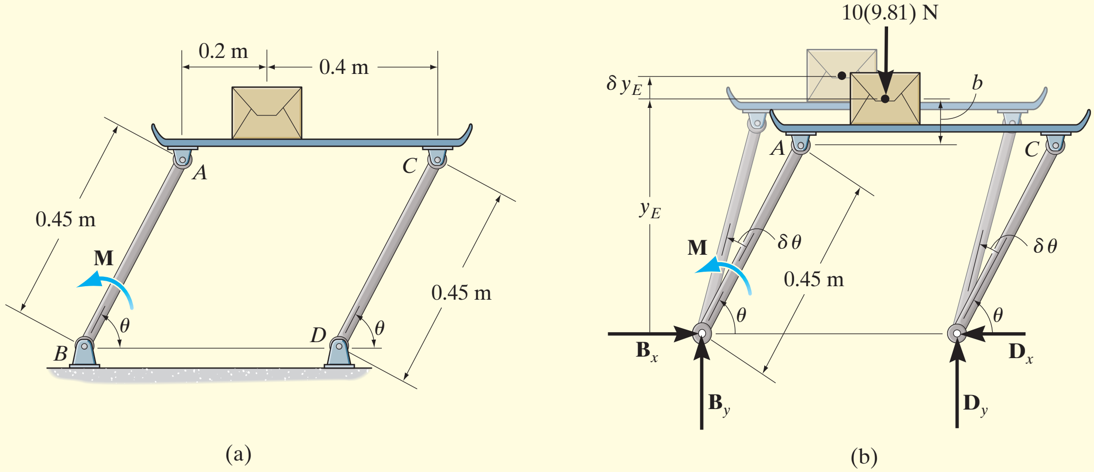


Fig. 11–8

SOLUTION

Free-Body Diagram. When θ undergoes a positive virtual displacement $\delta\theta$, only the couple moment **M** and the weight of the box do work, Fig. 11–8*b*.

Virtual Displacements. The position coordinate y_E, measured from the *fixed point* B, locates the weight, 10(9.81) N. Here,

$$y_E = (0.45 \text{ m}) \sin \theta + b$$

where b is a constant distance. Differentiating this equation, we obtain

$$\delta y_E = 0.45 \text{ m} \cos \theta \, \delta\theta \qquad (1)$$

Virtual-Work Equation. The virtual-work equation becomes

$$\delta U = 0; \qquad M\,\delta\theta - [10(9.81) \text{ N}]\delta y_E = 0$$

Substituting Eq. 1 into this equation

$$M\,\delta\theta - 10(9.81) \text{ N}(0.45 \text{ m} \cos \theta\, \delta\theta) = 0$$
$$\delta\theta(M - 44.145 \cos \theta) = 0$$

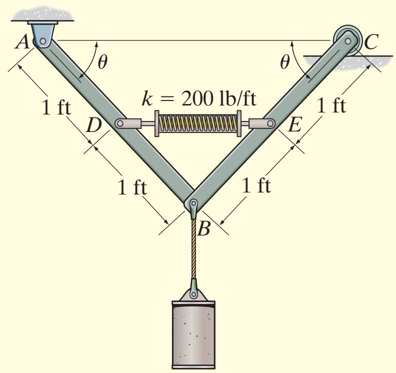
Since $\delta\theta \neq 0$, then

$$M - 44.145 \cos \theta = 0$$

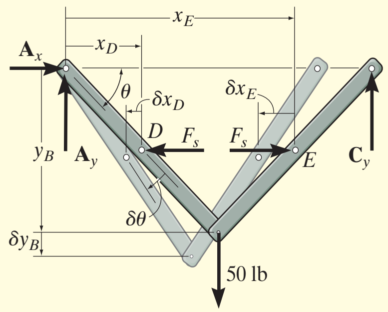
Since it is required that $\theta = 60°$, then

$$M = 44.145 \cos 60° = 22.1 \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

EXAMPLE 11.4



(a)



(b)

Fig. 11–9

The mechanism in Fig. 11–9*a* supports the 50-lb cylinder. Determine the angle θ for equilibrium if the spring has an unstretched length of 2 ft when $\theta = 0°$. Neglect the mass of the members.

SOLUTION

Free-Body Diagram. When the mechanism undergoes a positive virtual displacement $\delta\theta$, Fig. 11–9*b*, only $\mathbf{F}_s$ and the 50-lb force do work. Since the final length of the spring is 2(1 ft cos θ), then

$$F_s = ks = (200\text{ lb/ft})(2\text{ ft} - 2\text{ ft}\cos\theta) = (400 - 400\cos\theta)\text{ lb}$$

Virtual Displacements. The position coordinates x_D and x_E are established from the *fixed point* A to locate $\mathbf{F}_s$ at D and at E. The coordinate y_B, also measured from A, specifies the position of the 50-lb force at B. The coordinates can be expressed in terms of θ using trigonometry.

$$x_D = (1\text{ ft})\cos\theta$$

$$x_E = 3[(1\text{ ft})\cos\theta] = (3\text{ ft})\cos\theta$$

$$y_B = (2\text{ ft})\sin\theta$$

Differentiating, we obtain the virtual displacements of points D, E, and B as

$$\delta x_D = -1\sin\theta\,\delta\theta \quad (1)$$

$$\delta x_E = -3\sin\theta\,\delta\theta \quad (2)$$

$$\delta y_B = 2\cos\theta\,\delta\theta \quad (3)$$

Virtual-Work Equation. The virtual-work equation is written as if all virtual displacements are positive, thus

$$\delta U = 0; \qquad F_s\,\delta x_E + 50\,\delta y_B - F_s\,\delta x_D = 0$$

$$(400 - 400\cos\theta)(-3\sin\theta\,\delta\theta) + 50(2\cos\theta\,\delta\theta)$$

$$-(400 - 400\cos\theta)(-1\sin\theta\,\delta\theta) = 0$$

$$\delta\theta(800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

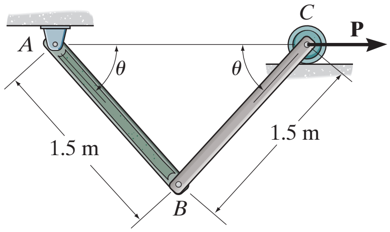
$$800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta = 0$$

Solving by trial and error,

$$\theta = 34.9° \qquad \textit{Ans.}$$

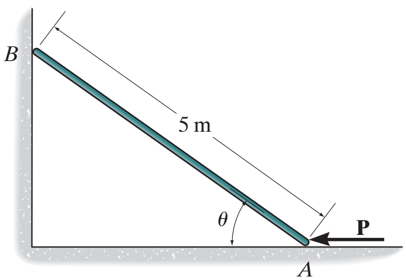
FUNDAMENTAL PROBLEMS

F11–1. Determine the required magnitude of force **P** to maintain equilibrium of the linkage at $\theta = 60°$. Each link has a mass of 20 kg.



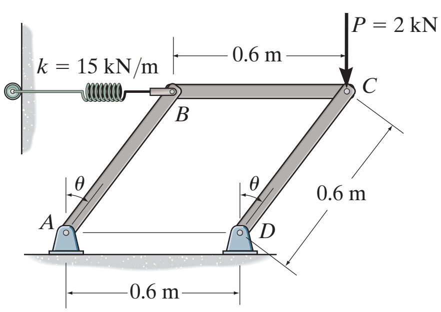
Prob. F11–1

F11–2. Determine the magnitude of force **P** required to hold the 50-kg smooth rod in equilibrium at $\theta = 60°$.



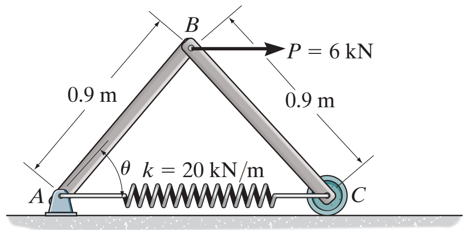
Prob. F11–2

F11–3. The linkage is subjected to a force of $P = 2$ kN. Determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Neglect the mass of the links.



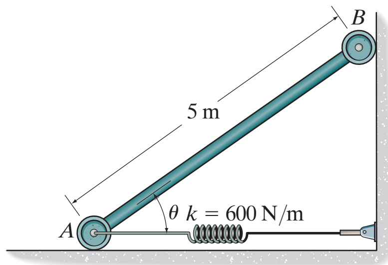
Prob. F11–3

F11–4. The linkage is subjected to a force of $P = 6$ kN. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 60°$. Neglect the mass of the links.



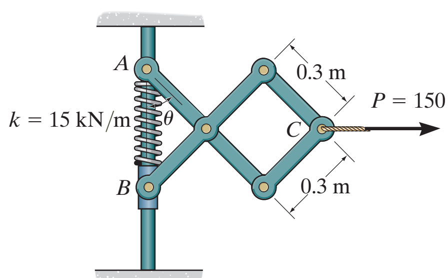
Prob. F11–4

F11–5. Determine the angle θ where the 50-kg bar is in equilibrium. The spring is unstretched at $\theta = 60°$.



Prob. F11–5

F11–6. The scissors linkage is subjected to a force of $P = 150$ N. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 0°$. Neglect the mass of the links.



Prob. F11–6